2024 HAWAII UNIVERSITY INTERNATIONAL CONFERENCES
ARTS, HUMANITIES, SOCIAL SCIENCES, & EDUCATION JANUARY 4 - 6, 2024
PRINCE WAIKIKI RESORT, HONOLULU, HAWAII

INSURANCE PRICING, REINSURANCE AND INVESTMENT DECSION OF INVESTMENT LINKED INSURANCE PRODUCTS BASED ON THE MUTUAL BENEFIT OF THE INSURER AND THE CONSUMER

MAO, HONG
SHANGHAI SECOND POLYTECHNIC UNIVERSITY

OSTASZEWSKI, KRZYSZTOF M.
DISTINGUISHED PROFESSOR OF MATHEMATICS
ACTUARIAL PROGRAM DIRECTOR
ILLINOIS STATE UNIVERSITY
NORMAL, ILLINOIS

WANG, JIN
OFFICE OF INFORMATION SECURITY OF CHINA

Prof. Mao Hong
Shanghai Second Polytechnic University

Prof. Krzysztof M. Ostaszewski
Distinguished Professor of Mathematics and Actuarial Program Director
Illinois State University
Normal, Illinois, USA

Mr. Jin Wang
Office of Information Security of China

Insurance Pricing, Reinsurance and Investment Decision of Investment Linked Insurance Products Based on the Mutual Benefit of the Insurer and the Consumer

Synopsis:

In this article, we establish an optimal decision model of insurance pricing, reinsurance and investment based on the mutual benefits of the insurer and the consumer. We assume that the price, investment and the claim loss rate are dependent stochastic processes. The main objective of our model is to maximize the product of expected utility of the consumer and to the expected utility of the terminal wealth of the insurer.

Insurance Pricing, Reinsurance and Investment Decision of Investment Linked Insurance Products Based on the Mutual Benefit of the Insurer and the Consumer

Abstract

In this article, we establish an optimal decision model of insurance pricing, reinsurance and investment based on the mutual benefits of the insurer and the consumer. We assume that the price， investment and the claim loss rate are dependent stochastic processes. The main objective of our model is to maximize the product of expected utility of the consumer and to the expected utility of the terminal wealth of the insurer. We construct a HJB equation and determine the optimal price of the insurance products, the optimal reinsurance strategy, and the optimal investment portfolio of the insurer simultaneously by solving HJB equation. Finally, we carry out sensitivity analysis.

Key words：Mutual benefit；Insurance Pricing；Investment and reinsurance; HJB equation

1. Introduction

The investment decision of the insurer has been widely discussed in the literature. See papers: Browne (1995), Hipp and Plum (2000), Liu and Yang (2004), Browne (2000), Korn (2005), Korn and Seifried (2009), Mataramvura and Âksendal (2008), Promislow and Young (2005). All these papers discuss how to find the optimal investment portfolio of the insurers on the assumption that the price of the insurance products is given. There are also a lot of papers discussing the decision of investment and reinsurance. Schmidli (2002) considered a classical model with drift and discuss the optimal decision on investment and reinsurance strategy by establishing the objective function of minimizing the ruin probability via solving Hamilton-Jacobi-Bellman Equation. Luo, Taksar and Tsoi (2008) consider a problem of optimal reinsurance and investment for an insurer whose surplus is governed by a linear diffusion. Their main goal is to find optimal investment and reinsurance strategy which minimizes the probability of ruin. Cao and Wan (2009) discuss the optimal proportional reinsurance and investment based on Hamilton-Jacob-Bellman equation. Edoli and Runggaldier (2010) study the optimization of reinsurance and investment. They assume that the arrival of a claim and the change of the price of the underlying asset(s) corresponds to a Poisson point process. The objective is to maximize the expected total utility. And as a special case, they also discuss the maximizing exponential utility functions whereby negative value of the risk process are penalized. Lin and Li (2011) considered an optimal reinsurance-investment problem of an insurer whose surplus process follows a jump-diffusion model. The dynamics of the risky asset are governed by a constant elasticity of variance model to incorporate conditional heteroscedasticity. The objective of the insurer is to choose an optimal insurance-investment strategy so as to maximize the expected exponential utility of terminal wealth. Eisenberg and Schmidli (2011) study optimal control of a classical risk model and its diffusion approximation. They assume that the individual claims are reinsured fully or partially, and they also assume the insurer is allowed to invest in a riskless asset with some constant interest rate. The objective is to minimize the discount capital injections. They find explicit optimal solution by solving HJB equation. Liu et al. (2013) study an optimal investment –reinsurance problem for an insurer who faces dynamic risk constraint in a Markvian regime-switching environment. Zhang and Siu (2009) used stochastic differential game to optimize the investment and reinsurance strategy with model uncertainty. Their approach assumed that the insurance company invests in a capital market index whose dynamics follow a geometric Brownian motion. The risk process of the company is governed by either a compound Poisson process or its diffusion approximation. The company can also transfer a certain proportion of the insurance risk to a reinsurance company by purchasing reinsurance. They formulated the optimal investment problem as two player, zero-sum, stochastic differential games between the insurance company and the market. Here the market is a "fictitious" player of the game and selects a real-world probability measure so as to Max-Min the expected utility of the insurance company's terminal wealth when the surplus process of the insurance company is a compound Poisson process. They also discussed the case of Min-Max the ruin probability in insurance company when the surplus process of the insurance company is approximated as a diffusion process. Mao et al. (2016) extended the work of Zhang and Sui (2009) by examining the minmax exponential utility function and assuming the risk process of the insurer is a Cramer-Lundberg process. Furthermore, they extend their approach by including n $(n \geq 1)$ risky assets instead of only one risky asset, since the investment portfolio including multiple risky assets and a risk-free asset is more realistic. Additionally, they discuss the optimal investment and reinsurance problem under uncertainty by examining the minmax optimization of expected power utility function.

The issue of optimal setting of insurance price has also been studied extensively. Taylor (1986) developed a theory according to which premium rate may be optimally determined taking into account the effects of competition. Emms (2005) discuss how to calculate the premium using optimal control theory by maximizing the terminal wealth of an insurer under a demand law. Emms (2007) extend his research to calculate the premium using dynamic programming by maximizing the expected utility of the terminal wealth of an insurer. Emms and Haberman (2009) determine the optimal premium strategy in a competitive market using a deterministic general insurance model. However, all their research does not consider the effect of investment strategy of the insurer on the price of the insurance products and the effect of the shift and volatility of the price on the investment strategy. In the paper (Mao et al. 2013), they discuss how to determine the optimal price and optimal proportion of risky investment simultaneously. In their models, they assume that the insurance price, the investment funds, and the claim loss rate are correlated stochastic processes, we consider the effect of different investment strategies on the price and the effect of the drift and volatility of the insurance price on the investment strategy of the insurer. But above study is from the perspective of the insurer. In fact, it is not conformed with the customer's benefit if using the objective of maximizing the expected utility of the terminal wealth of the insurer. Josa-Fombellida and Rincón-Zapatero (2008) studied funding and investment decision of stochastic defined benefit pension plan. They establish the objective of minimizing the cost of contribution and to maximizing the utility of final surplus, measured as the relative fund level respect to the mean salary. Mao and Ostaszewski (2011) consider the insurance pricing as a bargaining process between the insurer and the customer and determine the optimal investment and pricing strategy based on the mutual benefits of insurers and customer. In this paper, we extend Mao and Ostaszewski (2011) to considering reinsurance. Different from general pricing of investment linked products, we make the funds invested as a part of price and the investment return as the negative price, the insurer and the consumer mutually share the investment return and mutually bear investment and underwriting risk with the insurer. The main objective of our models is to maximize the product of expected exponential utility of the customer and the expected exponential utility of the terminal wealth of the insurer. We consider a risky asset and a risk-free asset, and we also consider the possible reinsurance. We establish the HJB equation and obtain the optimal strategy of investment, reinsurance, and insurance price by solving the HJB equation.

The remainder of the paper is organized as follows. In section 2, the insurance models are given. In section 3, the HJB equations are given and the optimal price, the optimal reinsurance strategy and investment portfolio of the insurer are determined. Section 4 give an example and make some sensitivity analysis. Section 5 concludes our paper.

2. The Insurance, Reinsurance, and Investment Models

Assume that the insurance company can invest its wealth in one risk-free asset and one kind of risky assets, which can be traded continuously over time, no transaction cost or taxes are involved in trading. Assume that the price processes of risky investments follow geometric Brownian motions generated by W_1, i.e., $S(t)$ satisfies the stochastic differential equation:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW_1(t), \qquad S(0) = s_0, \tag{1}$$

where μ is the return rate of risky investment and σ is the standard deviation of risky investment.

The price of the risk-free asset is assumed to evolve according to

$$dB(t) = rB(t)dt, \tag{2}$$

where r is the risk-free interest rate and $r < \mu_i$, $i = 1, 2, ...n$.

Let the insurance premium p (per unit exposure) is stochastic, and

$$dp(t) = p(t)(\mu_p dt + \sigma_p W_3(t)) \qquad (3)$$

where μ_p is the shift of the premium and σ_p is the coefficient of volatility of the premium.

We suppose that $p(t)$ is a non-negative measurable process adapted to the filtration F_t satisfying the condition:

$$\int_0^T p^2(t)dt < \infty, \quad \mathrm{P} - a.s \qquad (4)$$

Let the claim process be the following stochastic process (Promislow and Young (2005)):

$$dC(t) = p_1 dt - \sigma_D dW_2(t) . \qquad (5)$$

Then, the surplus process (not including investment) of insurance company is:

$$dR(t) = (p(t)dt + p(t)\sigma_p dW_3(t)) - p_1 dt + \sigma_D dW_2(t) , \qquad (6)$$

where p_1 is the average rate of claim loss, and σ_D is the standard deviation of claim loss rate.

Let $\{X(t) / t \in [0,T]\}$ be the wealth process of the insurance company and the insurer uses the self-financing strategy. Let the amount of risky assets is $\pi(t)$ and the proportion of retention is $a(t)$. The wealth process of insurance company can be expressed as the following stochastic differential equation:

$$dX(t) = [a(t)(p(t) - p_1) + (\mu - r)\pi(t) + X(t)r]dt + \pi(t)\sigma dW(t)_1 + a(t)\sigma_D dW_2(t) + p(t)\sigma_p dW_3(t) ,$$

$$X(0) = x \qquad (7)$$

where $\{W_1(t), W_2(t) / t \geq 0\}$ and $\{W_1(t), W_3(t) / t \geq 0\}$ are two dependent standard Brownian motions on a filtered probability space , F_t is the P -augmentation of the natural filtration, W_1 , W_2 W_3 are one dimensional.

We suppose $\pi(t)$ and $a(t)$ are measurable control processes adapted to the filtration F_t, Markovian and stationary, satisfying

$$\int_0^T \pi^2(t)dt < \infty, \quad \mathrm{P}-a.s. \qquad (8)$$

and $\int a^2(t)dt < \infty$, P-$a.s$.

We assume that $\pi(t)$ is allowed to less than zero, that is, the short selling is permitted.
Assume that the investment is independent of claim loss, but the insurance price and claim loss is positively correlated between each other. The instant correlative coefficient is ρ_1. Assume that the insurance price and risky investment is negatively correlated between each other. The instant correlative coefficient is ρ_2.

3. HJB Equation and Solutions of Optimal Investment, Optimal Price of Insurance and Optimal Retention Proportion

We suppose that the utility function of the customer is $U_1(t,\ p(t)) = e^{-\alpha p}$, α $(0 < \alpha < 1)$ is the bargaining power of the consumer and the bargaining power of the insurer is $(1-\alpha)$. We also suppose that the utility function of the insurer is $U_2(X(t),\ p(t)) = e^{(1-\alpha)X}$.

We formulate the problem to maximize the product of the expected utility of the customer and the expected utility of the terminal wealth of the insurer. Given initial values of time, t_0, the wealth of the insurer, X_0, the objective functional over the class of admissible controls $\mathrm{A}_{t_0,\ X_0}$ is given by

$$J((t_0 = 0,\ X_0 = x);(\pi, a, p)) = E_{t_0=0,\ X_0=x}\left(U(X(T), p)\right) \qquad (9)$$

The optimal problem can be expressed as to find the value function $V(t,\ X)$ and optimal solutions of $(\pi, a, p) \in \mathrm{A}_{t_0, X_0, p_0}$, which satisfies with

$$V(t, X, p) = \sup_{(\pi,a,p)\in \mathrm{A}_{t,X,p}} \left(J(t, X, p);(\Lambda, p)\right) \qquad (10)$$

It is not difficult to show that $V(t,\ X,\ p)$ is a Markov process. For any twice continuously differential function $h \in C^{1,2}(O) \bigcap C(\overline{O})$, where

$O := (0,T) \times (0,\infty) \times ....(0,\infty)$ and $\overline{O}$ denotes the closure of O, there exits partial differential operator $L^{\pi,p,a}[h(x, \quad p)]$:

$$L^{\pi,p,a}[h(x,p)] = \frac{\partial h}{\partial t} + [a(t)(p(t)-p_1) + \pi(\mu - r) + rX(t)]\frac{\partial h}{\partial x}$$
$$+\frac{1}{2}(a^2\sigma_D^2 + \pi^2\sigma^2)\frac{\partial^2 h}{\partial x^2} + (\rho_1 p\sigma_p a\sigma_D + \rho_2 p\sigma_p \sigma\pi)\frac{\partial^2 h}{\partial x \partial p} + \mu_p p(t)\frac{\partial h}{\partial p} + \frac{1}{2}p^2(t)\sigma_p^2\frac{\partial^2 h}{\partial p^2} \tag{11}$$

It is not difficult to get the following verification theorem.

Theorem 1: Suppose that there exists a function $\phi(t, \quad x, \quad p) \in C^{1,2}(O) \bigcap C(\overline{O})$ and a Markov control $(\pi^*, a^*, p^*) \in \mathrm{A}$ such that

1. $L^{\pi,p,a}\left(\phi(t,x,p)\right) \le 0$ for all $(\pi, a, p) \in \mathrm{A}$ and $(t, \quad x, \quad p) \in O$;
2. $L^{\pi^*,p^*,a^*}\left(\phi(t,x,p)\right) = 0$ for all $(t, \quad x) \in O$;
3. for all $(\pi, a, p) \in \mathrm{A} : \lim_{t \to T^-} \phi(t,x,p) = e^{-\alpha p + (1-\alpha)x}$;

Then $\phi(t,x,p) = V(t,x,p)$, and (π^*, a^*, p^*) is an optimal (Markov) control.

In order to obtain the optimal value function $V(t, \quad X, \quad p)$ and the optimal control (π^*, a^*, p^*), we only need to solve the following HJB equation:

$$\begin{cases} \sup\limits_{(\pi,p,a)\in A} L^{\pi,p,a}\left(V(t,X,p)\right) = 0 \\ V(T,x,p) = -e^{-(\alpha(-p)+(1-\alpha)x)} \end{cases}, \tag{12}$$

To solve the above HJB equation, we try to find a solution of the following form:

$$\phi^1(t, \quad x \quad p) = -e^{-(-\alpha p + (1-\alpha)x)e^{r(T-t)} + f(t))}, \tag{13}$$

where $f(t)$ is a undetermined function and $f(T)$ =0.

Substituting the above trial function into equation (11) yields:

$$L^{\pi,a,p}[\phi^1(t,x,p)]$$
$$=\phi^1(t,x,p)\times\{f_t-(p(a(1+\eta)-\eta)-\alpha pr/(1-\alpha)-ap_1+\pi(\mu-r))(1-\alpha)e^{r(T-t)}$$
$$+\frac{1}{2}(a^2\sigma_D^2+\pi^2\sigma^2)(1-\alpha)^2e^{2r(T-t)}-p\sigma_p(\rho_1 a\sigma_D+\sigma\rho_2\pi)(a(1+\eta)-\eta)\alpha(1-\alpha)e^{2r(T-t)}$$
$$+\mu_p p(a(1+\eta)-\eta)\alpha e^{r(T-t)}+\frac{1}{2}p^2\sigma_p^2(a(1+\eta)-\eta)^2\alpha^2e^{2r(T-t)})\} \quad (14)$$

By maximizing over (π,p,a) yields the following first order condition for the maximum point $(\hat{\pi}\ \hat{p},\hat{a})$:

$$\hat{p}=\frac{A+B-C}{D},$$

where
$$A=\frac{1-(1+\mu_p)\alpha+\rho_1\sigma_p\sigma_D a\alpha(1-\alpha)e^{r(T-t)}}{\sigma_p^2\alpha^2(a(1+\eta)-\eta)e^{r(T-t)}},\ B=\frac{\rho_2(\mu-r)}{\sigma_p\sigma(a(1+\eta)-\eta)\alpha e^{2r(T-t)}},$$
$$C=\frac{r}{\sigma_p^2(a(1+\eta)-\eta)^2\alpha e^{r(T-t)}}$$

and $D=1-\dfrac{\rho_2^2}{\alpha e^{r(T-t)}}$, (15)

$$\hat{\pi}=\frac{\mu-r}{\sigma^2(1-\alpha)e^{r(T-t)}}+\frac{(A+B-C)\rho_2\sigma_p(a(1+\eta)-\eta)\alpha}{D\sigma(1-\alpha)} \quad (16)$$

$$\frac{\partial p}{\partial a}=\frac{\frac{\partial A}{\partial a}+\frac{\partial B}{\partial a}-\frac{\partial C}{\partial a}}{D} \quad (17)$$

where

$$\frac{\partial A}{\partial a}=\frac{\rho_1\sigma_p\sigma_D\alpha(1-\alpha)e^{r(T-t)}(a(1+\eta)-\eta)-\sigma_p^2\alpha^2(1+\eta)\left(1-(1+\mu_p)\alpha+\rho_1\sigma_p\sigma_D a\alpha(1-\alpha)\right)}{\sigma_p^2\alpha^2(a(1+\eta)-\eta)^2e^{r(T-t)}}$$

$$\frac{\partial B}{\partial a}=-\frac{\rho_2(\mu-r)}{\sigma_p^2\alpha(a(1+\eta)-\eta)^2e^{2r(T-t)}}$$

$$\frac{\partial C}{\partial a}=-\frac{r(1+\eta)}{2\sigma_p^2\alpha(a(1+\eta)-\eta)^3e^{r(T-t)}}$$

$$\frac{\partial\pi}{\partial a}=\frac{\partial p}{\partial a}\cdot\frac{\rho_2\sigma_p(a(1+\eta)-\eta)\alpha}{\sigma(1-\alpha)}+\frac{(A+B-C)\rho_2\sigma_p(1+\eta)\alpha}{D\sigma(1-\alpha)} \quad (18)$$

$$\left(-\frac{\partial p}{\partial a}(a(1+\eta)-\eta)-p(1+\eta)+\frac{\partial p}{\partial a}\alpha r/(1-\alpha)+p_1-(\mu-r)\frac{\partial \pi}{\partial a}\right)(1-\alpha)$$
$$+\left(a\sigma_D^2+\pi\sigma^2\frac{\partial\pi}{\partial a}\right)(1-\alpha)^2e^{r(T-t)}$$
$$-\left(\left(\frac{\partial p}{\partial a}\sigma_p(\rho_1 a\sigma_D+\rho_2\sigma\pi)+p\sigma_p\left(\rho_1\sigma_D+\rho_2\sigma\frac{\partial\pi}{\partial a}\right)\right)(a(1+\eta)-\eta)+p\sigma_p(\rho_1 a\sigma_D+\rho_2\sigma\pi)(1+\eta)\right)\alpha(1-\alpha)e^{r(T-t)}$$
$$+\mu_P\left(\frac{\partial p}{\partial a}(a(1+\eta)-\eta)+p(1+\eta)\right)\alpha+\sigma_p^2\left(p\frac{\partial p}{\partial a}\left(a(1+\eta)-\eta\right)^2+p^2(a(1+\eta)-\eta)(1+\eta)\right)\alpha^2e^{r(T-t)}=0$$
(19)

Putting equations (15), (16) ,(17) and (18) into (14), we get the optimal solution of $\hat{a}$, if $\hat{a}\le 1$, put the value of $\hat{a}$ into equations (15) and (16) and find the optimal solutions of $\hat{\pi}$ and $\hat{p}$, otherwise, set $\hat{a}=1$ and obtain the optimal solutions of $\hat{\pi}$ and $\hat{p}$ by solving equations (15) and (16).

Substituting the optimal solutions of $(\hat{\pi}\ \hat{p},\hat{a})$ into (14) and setting it equal to 0, we obtain

$$f_t=(\hat{p}(\hat{a}((1+\eta)-\eta)-p_1+\alpha\,\hat{p}\,r/(1-\alpha)+(\mu-r))(1-\alpha)e^{r(T-t)}$$
$$-\frac{1}{2}(\hat{a}^2\sigma_D^2+\pi^2\sigma^2)(1-\alpha)^2e^{2r(T-t)}$$
$$+(\rho_1\,\hat{p}\,\sigma_p\,\hat{a}\,\sigma_D+\rho_2\,\hat{p}\,\sigma_p\pi\sigma)(\hat{a}((1+\eta)-\eta)\alpha(1-\alpha)e^{2r(T-t)}$$
$$-\mu_p\,\hat{p}(\hat{a}((1+\eta)-\eta)\alpha e^{r(T-t)}-\frac{1}{2}\hat{p}^2(\hat{a}((1+\eta)-\eta)^2\sigma_p^2\alpha^2e^{2r(T-t)}),$$
(20)

where $(\hat{\pi},\hat{p},\hat{a})$ satisfy with equations (15), (16), (17), (18) and (19) respectively.
Therefore, according to the above analysis, we get the following theorem.
Theorem 2: when the expected utility function is power function, the optimal strategy $(\pi^*,\ p^*,\ a^*)=(\hat{\pi},\ \hat{p},\ \hat{a})$ is given by equation (15), (16), (17), (18) and (19) and the optimal value function is:

$$V(t,\ x,\ p)=e^{(-\alpha p+(1-\alpha)x)e^{r(T-t)}+f(t)}, \qquad (21)$$

where $f(t)$ is given by equation (20).

In the following we will analysis the relationship between optimal solutions and all parameters using an example.

4. An example and numerical analysis

Assume that $\sigma_D = 0.15, r = 0.05, \alpha = 0.50, \mu = 0.1, \sigma = 0.1, p_1 = 0.1, \sigma_p = 0.15$
$\mu_p = 0.05, \alpha = 0.50, T = 20, \rho_1 = 0.20, \rho_2 = -0.2$

With the help of Matlab software, we obtain optimal solutions when time t=1,2, …, 20 shown in Table 1:

Table1. Optimal solutions when time takes different values:

t	1	2	3	4	5	6	7	8
$a^*(t)$	0.1875	0.1876	0.1877	0.1877	0.1878	0.1878	0.1879	0.1880
$p^*(t)$	0.2347	0.2115	0.1835	0.1500	0.1101	0.0631	0.0079	-0.0563
$\pi^*(t)$	3.8637	4.0623	4.2712	4.4909	4.7219	4.9648	5.2203	5.4890
t	9	10	11	12	13	14	15	16
$a^*(t)$	0.1881	0.1881	0.1882	0.1883	0.1884	0.1885	0.1885	0.1886
$p^*(t)$	-0.1309	-0.2172	-0.3166	-0.4309	-0.5619	-0.7117	-0.8828	-1.0777
$\pi^*(t)$	5.7716	6.0688	6.3814	6.7101	7.0559	7.4196	7.8022	8.2047
t	17	18	19	20				
$a^*(t)$	0.1887	0.1888	0.1889	0.1890				
$p^*(t)$	-1.2995	-1.5516	-1.8377	-2.1621				
$\pi^*(t)$	8.6281	9.0734	9.5420	10.0350				

From Table 1, we find that optimal retention change little with the change of the time, optimal price decreases with time and optimal risky assets increases with time. From Table 1, we also see that optimal price is rather greater at the start stage, but it goes negative with the increase of time. The possible explanation may be that here the price includes claim loss payment in future and at the same time, includes investment funds prepaid by the consumer. The negative price means that from 8-th year, the consumer will not pay premium, but obtain the net investment return offsetting the premium to instead. In the following, we will do sensitivity analysis.

4.1. Varying the parameters of α, μ_p and σ_p

Figure 1(a) through Figure 1(i) display the change patterns of optimal insurance price, p, optimal retention proportion, a, and optimal amount of risky assets, π, when the parameters of α, μ_p, σ_p change. From Figure 1(a) through Figure 1(c), we find that bargaining power of the consumer, α, greatly affect optimal insurance price, The greater the value of α is, the lower the optimal price it is. However, the effect of α on the optimal retention proportion is relatively smaller than that on the optimal price. Increasing the value of α will increase the optimal proportion of retention. That means that the insurer tends to increase retention proportion when the bargaining power of the consumer is higher in order to have more funds to invest. Finally, we find that increasing bargaining power of the consumer will make the investment more aggressive in order to make more investment return to compensate loss due to lower price.

We find from Figure 1(d) that the higher the shift of the insurance price, μ_p, the lower the optimal insurance price it is. However, the optimal insurance price is very insensitive to the change of μ_p. From Figure 1 (e) we find that optimal retention proportion is positively related to μ_p, but it is not very sensitive to the change of μ_p. Finally, we find from Figure 1 (f) that the optimal amount of risky assets is very insensitive to the change of μ_p.

Figure 1(j) through Figure 1(l) show that the volatility of insurance price, σ_p, has some effect on optimal price and increasing σ_p will increase the optimal price. However, it has little effect on optimal retention proportion and optimal amount of risky assets.

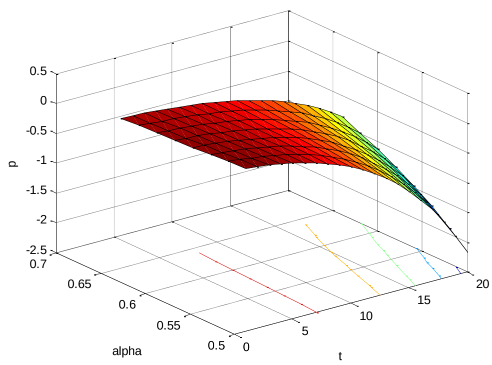

Figure 1(a). The change pattern of optimal insurance price with bargaining power of the consumer and time

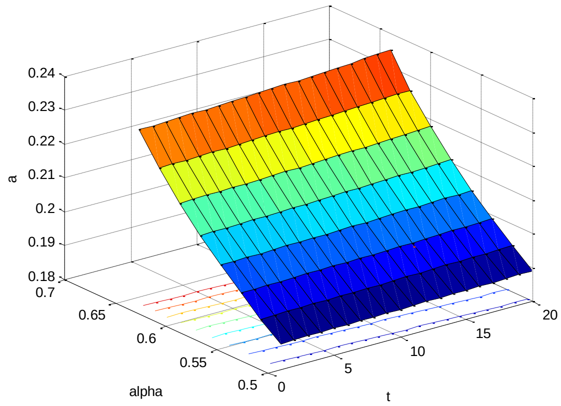

Figure 1(b). The change pattern of optimal retention proportion with bargaining power of the consumer and time

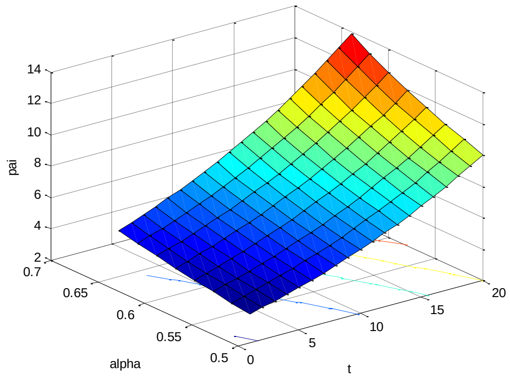

Figure 1(b). The change pattern of optimal amount of risky assets with bargaining power of the consumer and time

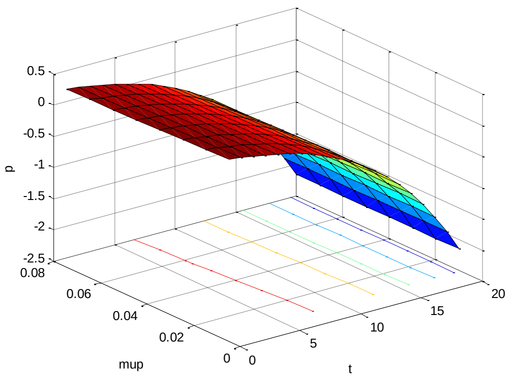

Figure 1(d). The change pattern of optimal insurance price with shift of insurance price and time

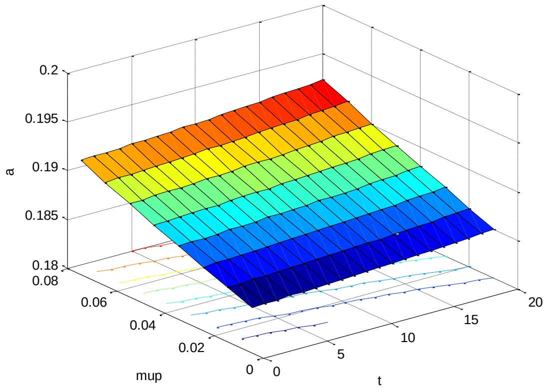

Figure 1(e). The change pattern of optimal retention proportion with shift of insurance price and time

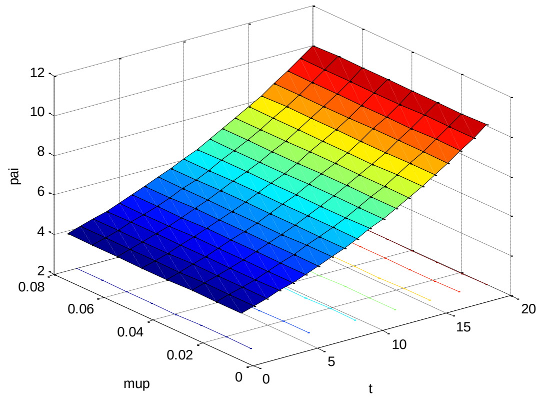

Figure 1(f). The change pattern of optimal amount of risky assets with shift of insurance price and time

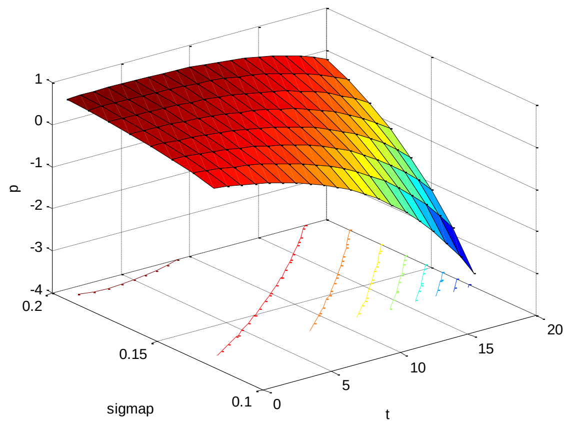

Figure 1(g). The change pattern of optimal insurance price with volatility of insurance price and time

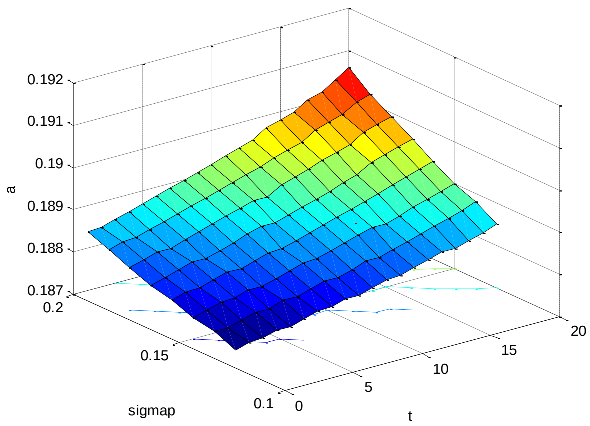

Figure 1(h) The change pattern of optimal retention proportion with volatility of insurance price and time

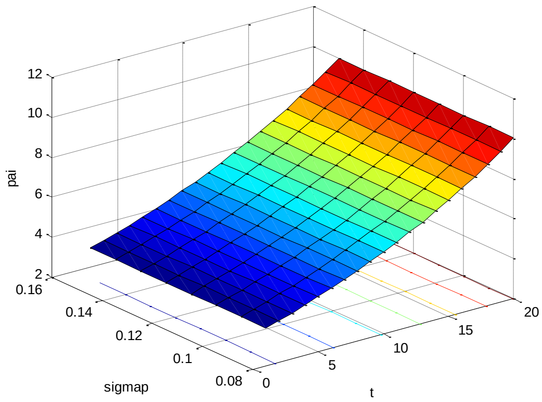

Figure 1(i). The change pattern of optimal amount of risky assets with volatility of insurance price and time

4.2. Varying parameters μ, σ and r

Figure 2(a) through Figure 2(c) show that the optimal price and optimal amount of risky assets are very sensitive to the means of return rate of risky assets μ. Increasing μ will greatly decrease the optimal price and increase the optimal risky assets. However, the optimal retention is insensitive to the change of μ. It should be noticed that sensitivity of optimal price to time increases with the increase of μ and the sensitivity of optimal amount of risky assets increases with the increase of μ.

Figure 2 (d) through (f) show that the optimal price and optimal retention are not sensitive to the change of volatility of the return rate of risky assets σ. Increasing σ

will slightly decrease optimal price and optimal retention proportion. However optimal amount of risky assets is very sensitive to the change of σ. Increasing σ will greatly decrease optimal risky assets.

From Figure 2 (g) through (i) we find the that increasing risk-free interest rate, r, will decrease the optimal insurance price. However, the sensitivity of optimal price with time decreases with the increase of r. We also find that increasing risk-free interest rate will rather greatly increase the optimal retention proportion. The possible explanation may be that the higher risk-free interest rate means that it is saver to both consumer and insurer in investment and it promotes insurer to decrease reinsurance. We also find that increasing r will decrease the optimal risky assets. And the sensitivity of optimal risky assets with time increases with the increase of r.

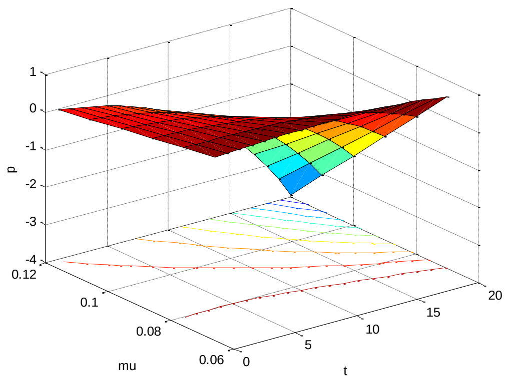

Figure 2(a) The change pattern of optimal insurance price with means of return rate of risky assets and time

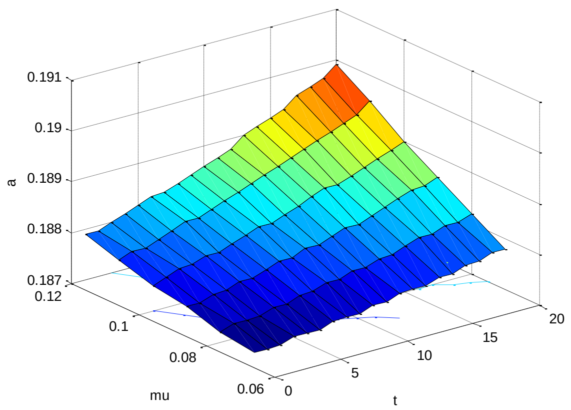

Figure 2(b) The change pattern of optimal retention proportion with means of return rate of risky assets and time

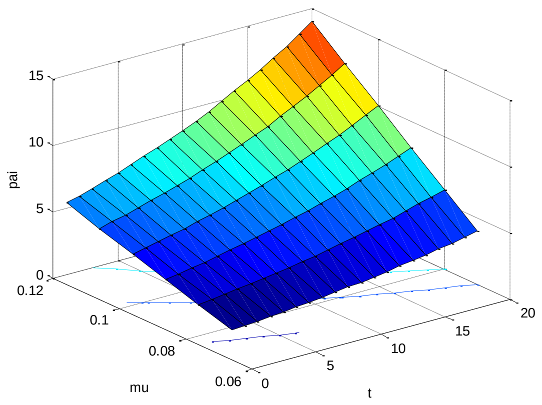

Figure 2(c) The change pattern of optimal amount of risk assets with means of return rate of risky assets and time

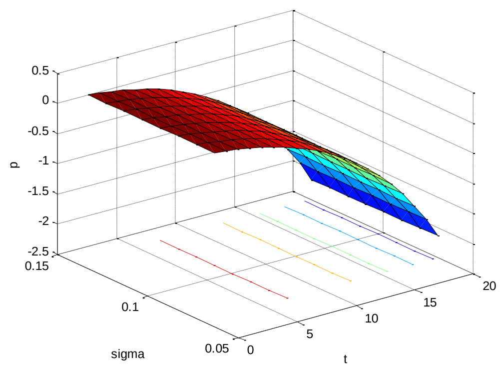

Figure 2(d) The change pattern of optimal insurance price with volatility of return rate of risky assets and time

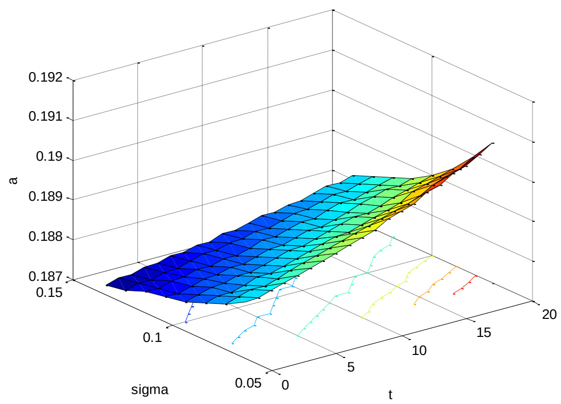

Figure 2(e) The change pattern of optimal retention proportion with volatility of return rate of risky assets and time

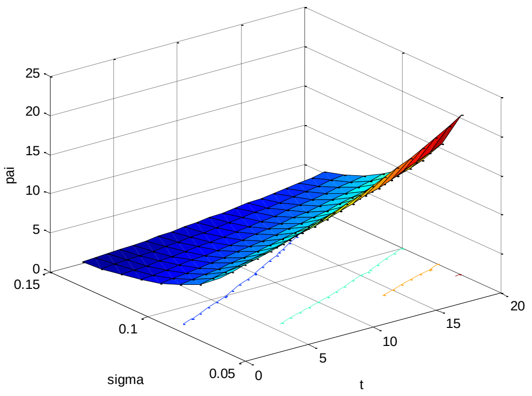

Figure 2(f) The change pattern of optimal amount of risky assets with volatility of return rate of risky assets and time

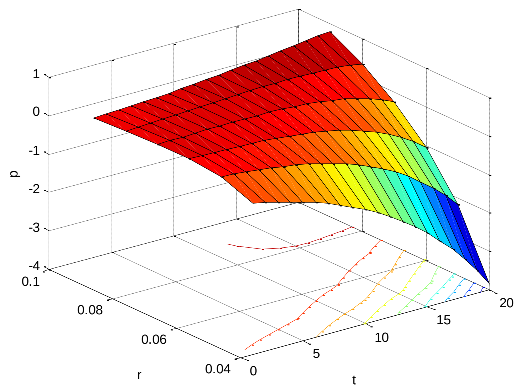

Figure 2(g) The change pattern of optimal insurance price with the risk-free interest rate and time

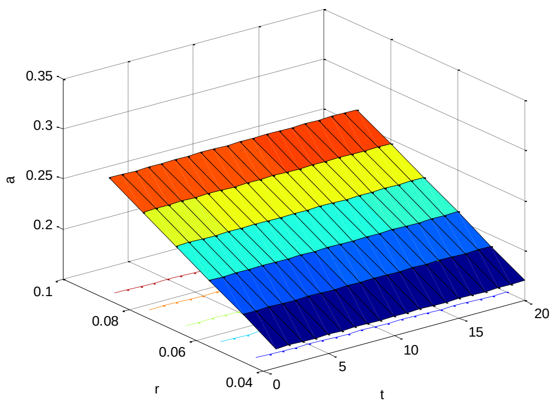

Figure 2(h) The change pattern of optimal retention proportion with the risk-free interest rate and time

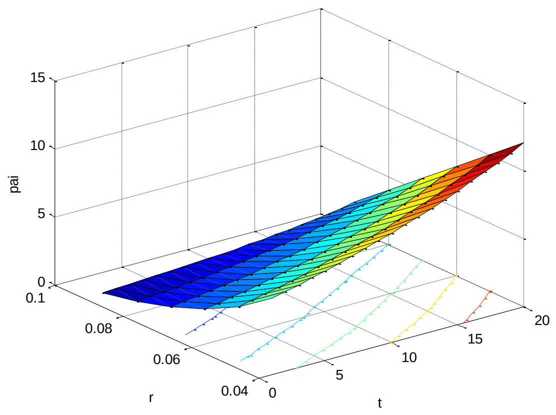

Figure 2(i) The change pattern of optimal amount of risky assets with the risk-free interest rate and time

4.3. Varying the parameters σ_D and η

Figure 3(a) through Figure 3(c) show that increasing the volatility of claim loss, σ_D will slightly decreases optimal insurance price, slightly increases optimal amount of risky assets, but has little effect on optimal retention proportion. It illustrates that higher volatility of claim loss means lower quality of underwriting products, it can only be underwritten at lower market price, however, insurer will appropriately increase the proportion of risky assets invested so as to gain more investment income and further to offset the decrease of underwriting income.

From Figure 3(e) through Figure 3(f), we see that the optimal price and optimal amount of risky assets is insensitive to the change the rate of reinsurance cost η. However, the optimal retention proportion is rather sensitive to the change of η. Increasing η will rather greatly increase the retention proportion.

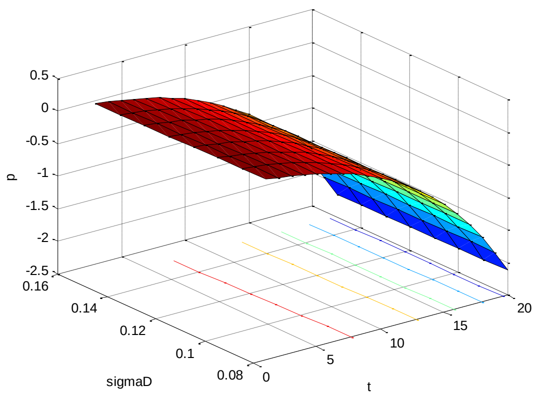

Figure 3(a) The change pattern of optimal insurance price with the volatility of claim loss and time

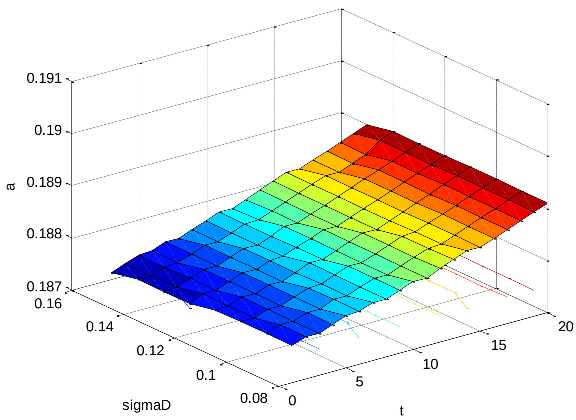

Figure 3(b) The change pattern of optimal retention proportion with the volatility of claim loss and time

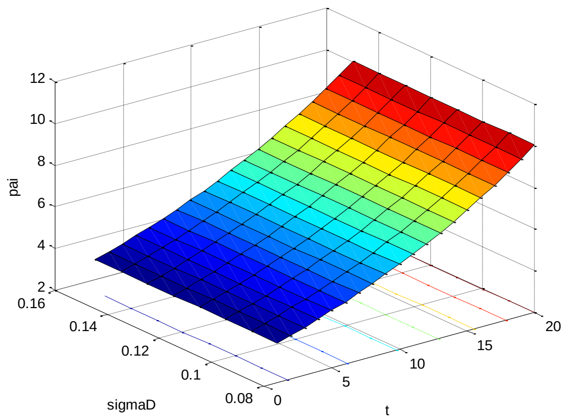

Figure 3(c) The change pattern of optimal amount of risky assets with the volatility of claim loss and time

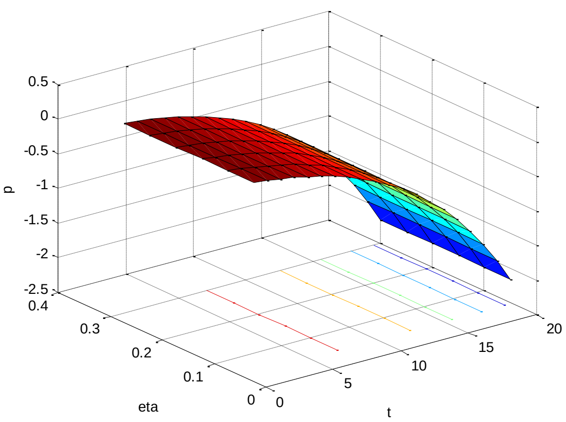

Figure 3(d) The change pattern of optimal insurance price with the rate of the reinsurance cost and time

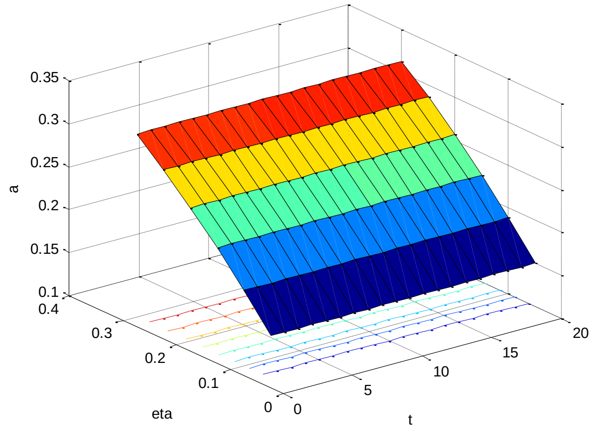

Figure 3(e) The change pattern of optimal retention proportion with the rate of the reinsurance cost and time

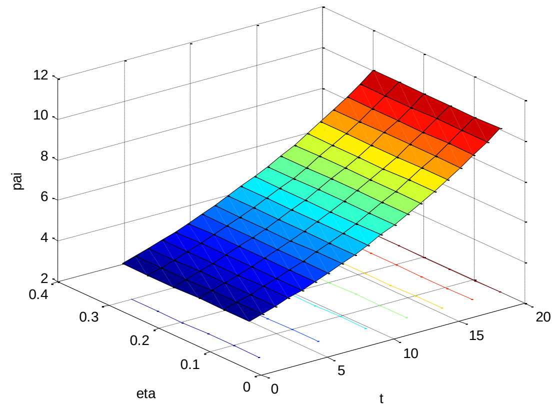

Figure 3(f) The change pattern of optimal amount of risky assets with the rate of the reinsurance cost and time

4.4. What happens when parameters ρ_1 and ρ_2 change

From Figure 4(a) through Figure 4 (c) we find that the optimal price decrease with the increase of correlation coefficient between price and claim loss ρ_1 and both of optimal retention proportion and optimal amount of risky assets are insensitive to the change of ρ_1. Figure 4(d) through Figure 4(f) show that the optimal insurance price is rather sensitive to the change of correlation coefficient between price and investment ρ_2. And the optimal price is negatively related to ρ_2. However, the optimal retention proportion and optimal risky assets are not sensitive to the change of ρ_2.

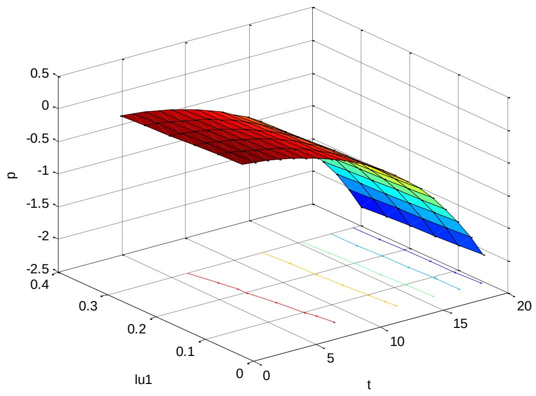

Figure 4(a) The change pattern of optimal insurance price with the correlation coefficient between price and claim loss and time

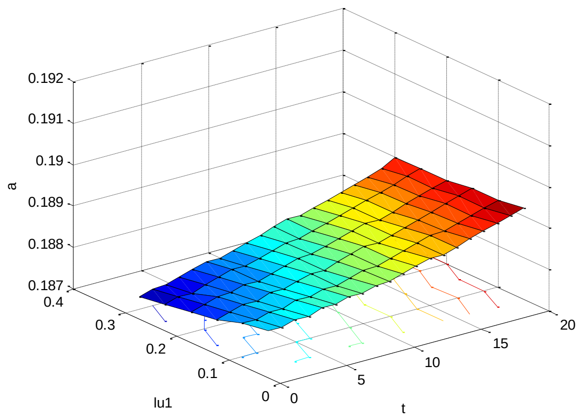

Figure 4(b) The change pattern of optimal retention proportion with the correlation coefficient between price and claim loss and time

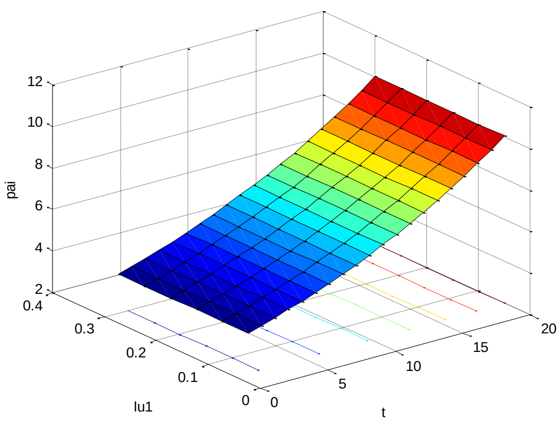

Figure 4(c) The change pattern of optimal amount of risky assets with the correlation coefficient between price and claim loss and time

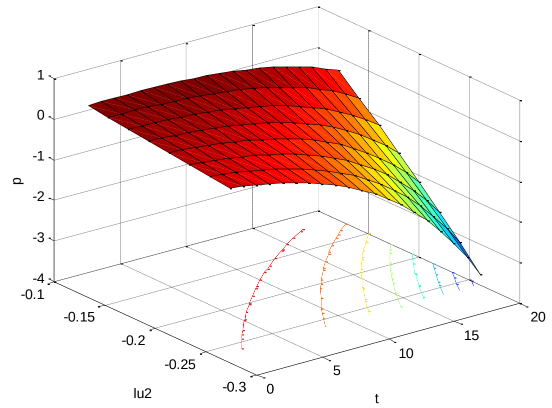

Figure 4(d) The change pattern of optimal insurance price with the correlation coefficient between price and risky assets and time

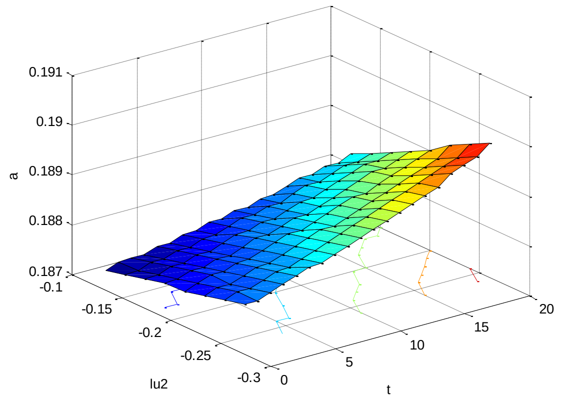

Figure 4(e) The change pattern of optimal retention proportion with the correlation coefficient between price and risky assets and time

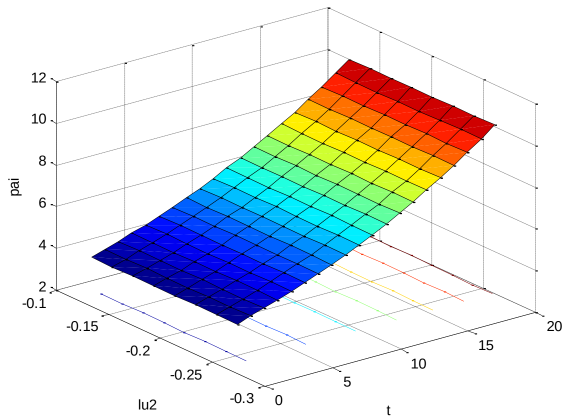

Figure 4(f) The change pattern of optimal amount of risky assets with the correlation coefficient between price and risky assets and time

Table2. The direction of the change of the optimal solutions when the parameters change:

parameters	$p*$	$a*$	$\pi*$
μ	(-)	(+) slightly	(+)
σ	(+) slightly	(-) slightly	(-)
μ_p	(-)	(+) slightly	(+) slightly
σ_p	(+)	(+) slightly	(-) slightly
σ_D	(-) slightly	almost no relation	(+) slightly
ρ_1	(-)	(-) slightly	almost no relation
ρ_2	(-)	(+) slightly	(+) slightly
α	(-)	(+)	(+)
r	(+)	(+)	(-)
η	(+) slightly	(+)	almost no relation
t	(-)	(+) slightly	(+)

(+) expresses the direction of the change of the optimal solution is the same as that of parameters, and (-) expresses the direction of the change of the optimal solution is opposite to that of parameters.

Conclusions

In this article, we discuss the optimal decision of insurance price, the proportion of reinsurance and investment portfolio based on the mutual benefits of the insurer and the customer. We suppose that the insurance price, the claim loss, and the investment are correlated stochastic processes. The main objective of our model is to maximize the product of the expected utility of the consumer and the expected utility of terminal wealth of the insurer in a bounded horizon. We construct a HJB equation and determine the optimal price of the insurance products, the optimal reinsurance strategy and the optimal investment portfolio of the insurer simultaneously by solving HJB equation. Finally, we use an example to illustrate its application and the sensitivity analysis is also carried out. The results of sensitivity analysis shows that the optimal price $p*$ is sensitive to the bargaining power of the consumer α, the drift of return rate of risky assets μ, the risk-free interest rate r, the volatility of insurance price σ_p, the correlation coefficient between price and claim loss ρ_1 and the correlation coefficient between price and investment ρ_2. However it is insensitive to the shift of insurance price μ_p, the volatility claim loss σ_D, and the volatility of return rate of risky assets σ. The optimal retention is sensitive to the parameters of the rate of reinsurance cost η and risk-free interest rate r and rather sensitive to the parameter of bargaining power of the consumer α, but it is insensitive to the other parameters. The optimal amount of risky assets is sensitive to the parameters of α, μ, σ and r but insensitive to other parameters.

References:

Browne, S., 1995, Optimal investment policies for a firm with a random risk process: Exponential utility and minimizing the probability of ruin. *Mathematics of Operations Research*, 20 (4), 937–958.

Browne, S., 2000, Stochastic differential portfolio games. *Journal of Applied Probability*, 37 (1), 126-147.

Cao, Y., and N.Wan, (2009), Optimal proportional reinsurance and investment base on Hamilton-Jacobi-Bellman equation. *Insurance: Mathematics and Economic*. 45, 157-162.

Edoli, E. and W. J. Runggaldier, 2010. On optimal Investment and reinsurance context with a point process market model, *Insurance: Mathematics and Economics* 47, 315-326.

Emms, P. 2007a. Dynamic Pricing of General Insurance in a Competitive Market. *ASTIN Bulletin* 37(1): 1–34.

———. 2007b. Pricing General Insurance with Constraints. *Insurance: Mathematics and Economics* 40: 335–55.

Emms, P., and S. Haberman, 2005, Pricing General Insurance Using Optimal Control Theory. *ASTIN Bulletin* 35(2): 427–53.

Eisenberg, J. and H. Schmidli, 2011. Optimal control of capital injections by reinsurance with a constant rate of interest, *Journal of Applied Probability* 48: 733-748.

Hipp, C., Plum, M., 2000, Optimal investment for insurers. *Insurance Mathematics and Economics* 27 (2), 215-228.

Josa-Fombellida, R., Rincón-Zapatero, J.P., 2008a. Funding and investment decisions in a stochastic defined benefit pension plan with several levels of labor-income earnings. *Computers and Operations Research* 35, 47-63.

Korn, R. 2005, Worst-case scenario investment for insurers, *Insurance: Mathematics and Economics*, 31, 1-11.

Korn, R., F. T. Seifried. 2009, A worst-case approach to continuous-time portfolio optimization. *Radon Ser. Comput. Appl. Math.* 8:327–345.

Lin, Xiang and Y. Li, 2011. Optimal reinsurance and investment for a jump diffusion risk process under the CEV model. *North America Actuarial Journal* 15:417-431.

Liu, C., Yang, H., 2004, Optimal investment for an insurer to minimize its probability of ruin. *North American Actuarial Journal*, 8 (2), 11-31.

Liu, J., K-F. C. Yiu, T. K. Siu and W-K. Ching, 2013. Optimal investment-reinsurance with dynamic risk constraint and regime-switching, *Scandinavian Actuarial*

Journal, 2013(4):1-23. DOI:10.1080/03461238.2011.602477
Luo, S., M. Taksar and A. Tsoi, 2008, On reinsurance and investment for larg insurance portfolios, *Insurance: Mathematics and Economics*, 42: 434-444.
Mao, H, J. Carson, K. Ostaszewski and Z. Wen, 2013, Optimal decision on dynamic insurance price and investment portfolio of an insurer, *Insurance: Mathematics and Economics*, 52: 359-369.
Mao, H., J. M. Carson, K. Ostaszewski, Y. Luo and Y. Wang, 2016, Determining the Insurer's Optimal Investment and Reinsurance Strategy Based on Stochastic Differential Game, *Journal of Insurance Issues*, 39,182-202.
Mao, H. and K.M. Ostaszewski, 2011 Insurance Pricing, Reinsurance and Investment Decision Based on the Mutual Benefit of the Insurer and the Customer, *Proceedings of the 2011 IEEE International Conference on Industrial Engineering and Engineering Management*, pp. 91-95.
Mataramvura, S., Âksendal, B., 2008, Risk minimizing portfolios and HJBI equations for stochastic differential games. *Stochastic An International Journal of Probability and Stochastic Processes*, 80 (4), 317-337.
Promislow, D.S., Young, V.R., 2005, Minimizing the probability of ruin when claims follow Brownian motion with drift. *North American Actuarial Journal*. 9 (3), 109–128.
Schmidli, H., 2002. On minimizing the ruin probability by investment and reinsurance, *Annals of Applied Probability* 12: 890-907.
Zhang, X., Siu,T.K., 2009, Optimal investment and reinsurance of an insurer with model uncertainty. *Insurance: Mathematics and Economics*, 45, 81-88.