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Restoring Relevance to Verifiability

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Abstract. Verifiability plays a central role in guaranteeing meaningfulness in the positivist program. We briefly consider some related known difficulties, and then we focus on Ayer's original and revised verification principles, which were criticized by several philosophers. Ayer emphasized that empirical hypotheses must be *relevant* to observations and experiences. We use **R**, the *logic of relevant implication* to analyze some of the counter examples that critics proposed to show that verification principles are inadequate for excising nonsensical parts from knowledge.

Keywords: Ayer's verifiability principle; Logic **R**; Logical positivism; Relevance; Vienna Circle;

Introduction

Thinkers connected to the Vienna Circle and logical positivism were most often assuming — tacitly or explicitly — that *logic is 2-valued*. A reason for this might be the impact of L. Wittgenstein's *Tractatus Logico-Philosophicus*, which was read and debated in the Circle. Wittgenstein himself was influenced by Bertrand Russell and the *Principia Mathematica* by N. A. Whitehead and B. Russell, which played a crucial role in making 2-valued logic widely known all around Europe. It is customary to credit the invention of 2-valued logic to mathematicians and philosophers who worked in the 19th century, but in the early 20th century, the development of the theory of 2-valued logic was still in its early stages.[1] For example, K. Gödel's completeness and incompleteness theorems date to the beginning of the 1930s, and the undecidability of the lower functional calculus was proved by A. Church (and independently by A. M. Turing) in 1936. The first formal investigations of modal logics were carried out by C. I. Lewis, and intuitionistic logic was formulated by A. Heyting in 1930. Given the dominance of 2-valued logic in the first half of the 20th century, it is natural that it was taken by philosophers — by default — to be the underlying logic when they wished to use formal tools in their investigations of scientific methods and theories.

The limitations of the *expressive power* of 2-valued logic were realized from the beginning. For instance, Frege explicitly warned against reading causality or modality into a formula of the form $A \supset B$ (in contemporary notation), that is, into "if A, then B." The completeness proof of propositional logic with respect to truth-value interpretations by E. Post firmly established the *extensional* character of 2-valued logic. Many philosophers who relied on 2-valued logic openly acknowledged that this logic had certain shortcomings in capturing the content of arguments that were expressed in

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[1]We mention some names to hint at whom we are thinking about from amongst 19th-century scholars: A. De Morgan, G. Boole, G. Frege, C. S. Peirce and D. Hilbert.

a natural language such as English. However, logic underwent significant expansions in the last 70 years or so; a sketch of this progress could take up a whole book. For this paper, we focus on one logic from the family of *relevance logics*, namely, on the logic of relevant implication, which is often denoted by **R**. This logic has a semantical interpretation on a par with the Kripke-style (or possible world) semantics of modal logics. Both semantics utilize *accessibility relations* that relate certain kinds of descriptions of the world such as *situations*, *states of affairs* or *possible worlds*.

Our goal in this paper is to examine — using the logic **R** — some of the counter examples that were designed with the intention to show that one or another version of a verification principle is untenable. Relevance logic did not exist when A. J. Ayer formulated his revised verifiability principle, thus, we can only conjecture that he would have considered using relevance logic instead of 2-valued logic. We will show that the counter examples that were proposed are defused when **R** is used, and thereby, Ayer's appeal to relevance in verification is vindicated to a certain extent — whether he would have liked the relevance logic **R** or not.

The structure of the rest of the paper is as follows. In Section 1, we set the context for verifiability, and we consider early attempts to explicate verification and some counter arguments. Section 2 recalls Ayer's revised verification principle and Church's powerful refutation of its usefulness (assuming 2-valued logic, of course). Then, in Section 3, we outline **R** as an axiom system together with its interpretation, and we appraise three counter examples translated into **R**. Finally, we draw some conclusions in the last section.

1. First attempts to exclude nonsense

Rudolf Carnap in his two-part essay [7, 8] attributes to Wittgenstein the idea that meaningfulness stems from connections to empirical facts [7, p. 422]. According to Carnap, Wittgenstein advocated for a comprehensive principle that would require that every sentence of a language be completely verifiable [8, p. 18]. Of course, Wittgenstein's forceful stance may have been inspired by others, for example, by the physicist E. Mach and B. Russell. In its crudest form, the verifiability principle is "the thesis that a sentence is meaningful if and only if it is verifiable, and that its meaning is the method of its verification" [7, p. 421]. It seems important to note that in the middle of the 1930s, there were a range of informal characterizations of verifiability, and equally notably, Carnap was completely clear about the need of ramification of the principle.[2]

Carnap was careful to acknowledge that the members of the Circle held different views and that the prevalent take on verifiability had changed. The statement that we quoted above simply refers to "verification" without clarifying what the term means. The rapid emergence of new physical theories around the top of the 19th century was undoubtedly a source of concern about what can and what cannot be verified. Furthermore, there is a more broad worry — not limited to the domain of science —

[2]In this section, we shall use the term "verifiability principle" for several requirements stated and put forward by philosophers. We have to point out though that not each author used exactly these words, and some authors used more than one phrase to express roughly the same idea.

about the verification of statements of various sorts in philosophy, in every-day life, etc. We recall three types of perplexing examples (which are also mentioned by Carnap).

(1) It is obvious that nobody living today can verify whether Martin Luther nailed his 95 theses to the door of the Castle Church in Wittenberg on October 31st, 1517 — in the sense as (almost) anybody can verify what (if anything) he or she holds in his or her hand. Similarly, it is easy to formulate a statement about the future such as "Tomorrow lightening will strike the famous Gateway Arch in St. Louis, MO" that is not verifiable as of now. Philosophers have been aware, for millennia, of the problem of determining the truth value of statements that pertain to the past or to the future. History is usually not considered to be a science, hence, claims about happenings during the Reformation in the 16th century might be of little interest to a logical positivist like Carnap. However, certain future statements are about the outcomes of experiments or measurements that will be made in the future. The example we gave from meteorology may be somewhat dull, but anybody who could precisely predict with a 24 hour leeway if lightening will strike at any 10 (or even 100) square foot location in North America would likely become a celebrity (if not an instant billionaire). At this time, such statements are not verifiable at all — not today, not a day after. This example is to underscore that certain statements about the future are crucial to the sciences, and verifiability has to be stated in a sufficiently refined manner to be able to account for how such statements are treated in the sciences.

(2) Carnap imagines [8, pp. 8–9] a situation in which participants have different capacities for perception; people are far from equal in this respect. The easiest cases to notice is differences in vision. A colorblind person suffering from tritanopia cannot distinguish blues and greens. If the statement to be verified is "This cluster of space-time-points is blue" then it is possible that a colorblind and a non-colorblind person will make different judgements in a situation. Carnap wants to highlight that in the design of a formal language for a science an agreement should be reached about the predicates that are deemed to be verifiable.[3] A more fundamental point, which we could attribute to Wittgenstein, is that if the language does not contain certain predicates, then everything that could be expressed using those predicates (but not by other means in the language) is excluded from consideration. As Wittgenstein said in [18, 5.6] "*The limits of my language* mean the limits of my world." To bring up an example that is closer to contemporary physics, we can say that if the language of physics does not contain any predicates that pertain to the detection of dark matter, then dark matter cannot be connected to empirical observations. Of course, when a science is not formalized, then it is not be a matter of *language design* which predicates are included in the informal language that is actually used.

(3) The most problematic (and perennial) question with respect to verification in the sciences is *establishing* and *ascertaining laws*. In their simplest form, laws are *universally quantified conditionals*. A very basic illustration (from mathematics) is the statement that "All squares are rectangles," which could be formalized as $\forall x\,(Sx \supset Rx)$ in 2-valued

[3]It is not always clear, even to scientists, what is within the boundaries of perception and what is not. Turing [17] appears to give some credibility to ESP (extrasensory perception) by attempting to defend thinking machines against arguments based on ESP. He also claims that "Unfortunately the statistical evidence, at least for telepathy, is overwhelming." [17, p. 453.]

predicate logic with some mnemonic choices for two unary predicates. In geometry, this (or some other) universally quantified statement may be *proved* using accepted ways of arguing; hence, a universal claim (assuming that it is correct) can be unconditionally justified, given a mathematical theory. This is in sharp contrast with claims such as "All celestial bodies orbit the earth." Assuming for a moment that we are in the time of Ptolemy and we are adherents of his theory, we would, nonetheless, need to make new and new observations for each celestial body. Depending on our general theory of the cosmos, this could involve infinitely many observations of infinitely many planets, stars, asteroids and black holes, etc.

By the 1930s, logicians saw that the quantifiers "all" and "some" ($\forall$ and $\exists$) are not reducible to truth functional connectives. Indeed, Wittgenstein in [18, 5.521] pointed out that "I separate the concept *all* from the truth-function. ¶ Frege and Russell have introduced generality in connexion with the logical product or the logical sum." It is natural to associate $\forall$ with sum (or the material conditional) and $\exists$ with product.[4] The usual renderings of the Aristotelian categorical statements in 2-valued predicate logic illustrate this link that is easily motivated by how the corresponding natural language sentences are used. Carnap saw that in order to design a language the sentences of which are verifiable, it is not sufficient to require that the predicates are verifiable. He stipulates [8, p. 17] that "A universal or existential sentence which is restricted to a finite field (as e.g. the sentences constructed with restricted operators in the languages I and II ...) can be transformed into a conjunction or a disjunction respectively and therefore has the same character as a molecular sentence." It is conceivable that some universal sentences can be restricted to a finite domain. For example, the often used sample sentence "All men are mortal," at any given moment when it is uttered, quantifies over a finite domain, though a very large and messy domain. However, the laws of physics might create a problem for such a language, if, for instance, the universe is believed to be infinitely expansive. We may also note that the 2-valued predicate logic of finite domains is not exactly what is known as 2-valued logic, which eschews any restriction to finite domains. Carnap (correctly) thought that on a finite domain of named objects quantified sentences are reducible to statements of propositional logic. First-order languages do not require that each object has a name but it is plausible to assume that in a (generalized first-order) language used in physics, each object has a name, which may be derived using the unique names of space-time points in a four-dimensional coördinate system.

Carnap considers an alternative approach as to how to deal with laws of the sciences. Schlick — following Wittgenstein — expressed a view according to which the laws of physics are not universal conditional statements; rather, laws are *rules of procedure* for a scientist to discover true statements [8, pp. 18–19]. If we were to take this idea seriously, then we would need to remove the laws of physics from a formal theory of physics or at least the laws would need to be given an exceptional standing within the theory. For example, the law-like statements could be kept in a separate repository, and used through instantiation. In the sense of 2-valued predicate logic, this arrangement would have little impact on *what* can be deduced. However, it could affect the status of

[4]The conditional $\supset$ is not the same as sum $\vee$ (i.e., disjunction), but $\supset$ is *essentially* like $\vee$, because $A \supset B$ is nothing more than $\neg A \vee B$, (i.e., not-A or B).

the deduced statements, because presumably, they would not be *certainly true*, rather, they would be merely *likely true* statements.[5]

Alfred J. Ayer published his book *Language, Truth and Logic* in 1936, the same year Carnap's two-part essay started to appear. Ayer's book was widely read and it elicited reactions that we will scrutinize; hence, we recall how he handled verifiability.[6]

As a first attempt to exclude bogus sentences from philosophy, Ayer says [3, p. 35] that "[t]he criterion which we use to test the genuineness of apparent statements of fact is the criterion of verifiability. We say that a sentence is factually significant to any given person, if, and only if, he knows how to verify the proposition which it purports to express—that is, if he knows what observations would lead him, under certain conditions, to accept the proposition as being true, or reject it as being false." Then he proceeds to distinguish *practical verifiability* and *verifiability in principle*. The former is verifiability that is available to a person in the present. "Verifiability in principle" could be identified with logical possibility; however, Ayer seems to think of "in principle" as a physical possibility, which might not have yet been realized.

Ayer also distinguishes between *weak verifiability* and *strong verifiability*. The latter is taken to mean that the truth of a sentence can be *established conclusively* in experience, whereas the former only implies that the statement has been *shown probable* by experience. Ayer was not building a formal language (in a way that Carnap did); he wasn't even giving precise definitions as it would be customary within a formal framework. However, he was well aware of various difficulties that typically emerge in connection to verifiability in certain cases. Carnap's paper and Ayer's book appeared in the same year. Both writings discuss problems that arise when one tries to verify a sentence about the past or a universal statement. Ayer does not consider, in the latter case, the limitation to a finite domain though. Instead, he assumes that general laws of the sciences are statements that are genuinely applicable to infinitely many objects, hence, no finite series of observations may establish the truth of such statements.

To steer away from the term "verifiability," Ayer talks about statements of fact. And he says, along the lines of weak verifiability, that the question to be asked is simply "Would any observations be relevant to the determination of its truth or falsehood?" [3, p. 38], and a negative finding indicates that the statement is *nonsensical*. He rephrases what he means by a statement not being nonsensical into the following form: If a statement S, in conjunction with other statements, implies an observational statement that is not implied by those other statements alone, then S is a genuine factual proposition. Ayer immediately illustrates [3, p. 39 *ff*.] how to apply his criterion to statements from "traditional philosophy," since a goal of his is "to show that philosophy, as a genuine branch of knowledge, must be distinguished from metaphysics." [3, p. 41]

We turn to two early criticisms of Ayer's verifiability principle in [14] and [4]. Since Ayer did not give formal (or even semi-formal) definitions, it was perhaps, easier to find a bone of contention in his book than in Carnap's paper. Clearly, the project of

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[5]It is not completely clear whether the laws could be applied in deductions of other laws. For instance, the universally quantified chain rule $\{\, \forall x\,(Ax \supset Bx), \forall y\,(By \supset Cy)\,\} \vdash \forall z\,(Az \supset Cz)$, which is a derived rule in predicate logic, could result in another law without instantiation.

[6]The 2nd edition of Ayer's book is [3], in which he added a twenty-two-page introduction. The latter includes a revised version of his verification principle, which we will scrutinize in the next section.

ridding philosophy of metaphysical claims on the basis that those claims lack meaning did not sit well with many practitioners of philosophy.

M. Lazerowitz appears to be familiar with logic, however, in [14], he disagrees with Ayer about verification deciding the meaningfulness of statements. Indeed, he deploys the notation of the *Principia Mathematica* to formalize verifiability, having introduced distinct classes of statements. His formula (utilizing contemporary symbols here) is as follows:[7]

$$\forall s \forall p \left(E(s, p) \supset ((\varphi(p) \supset \psi(s)) \wedge (\neg\varphi(p) \supset \neg\psi(s))) \right)$$

To sum up the idea, in order for a sentence to be subject to verification and to a subsequent judgement about significance (i.e., meaningfulness), the sentence has to be known beforehand to express a proposition. Then p is verifiable if and only if the sentence that expresses it is significant. Lazerowitz's argument seems to miss what the main target of the logical positivists is, namely, the exclusion of certain sentences from the language of philosophy. Lazerowitz does not address any of the examples Ayer gave, nor he analyzes any similar examples. The sentence "Ethiopians are heavier than $\sqrt{2}$" [14, p. 375] seems to be a step closer to nonsensicality than sentences in metaphysics. This sample sentence could be dismissed simply on the basis of a category mistake, because "Ethiopian" would belong to anthropology (a social science) but $\sqrt{2}$ is a number from mathematics. Then "heavier than" could not be used to combine these expressions into a sentence.[8]

Another philosopher who objected to Ayer's verifiability principle was I. Berlin who wanted to show that the principle either had to be abandoned or it had to be revised. An interesting contribution of [4] is that it completely discards the idea that verification should involve relevant experience. In section 3, we show that the use of *relevance logic* alleviates some of the concerns, which were raised against versions of the verification principle.

Berlin reiterates that there are certain kinds of statements the verification of which runs into problems. Beyond recalling that statements about the past might be problematic, he emphasizes first-person indexical sentences. Incidentally, his focus on indexicals such as "If I look up I shall observe a blue patch" [14, pp. 236–237] obscures the equivalence that he claims that exists between hypothetical sentences such as "If s then p" and universal sentences such as "All s is p." Indeed, two sentences like "If a philosopher is British, then he is an empiricist" and "All philosophers who are British are empiricists" (or, more concisely, "All British philosophers are empiricists") are often formalized by the same formula in predicate logic. However, the indexical does not lend itself to the transformation that introduces a universal quantifier and replaces indefinite noun phrases with plural ones.

At least for two decades prior to [4], positivists kept acknowledging that the verification of universal statements or general laws runs into a difficulty. Yet, Berlin devotes more than three pages to the discussion of universally quantified statements,

[7] s and p are variables over sentences and propositions, respectively. E stands for "expresses," whereas φ reads as "is verifiable" and ψ means "is literally significant."

[8] Of course, the sentence could also be viewed as having a hiatus at its end, namely, a unit of mass such as *hundredweight*. It may be unusual to compare the weight of a person to an irrational amount of mass, but the claim would be verifiable for any given person.

and therewith, to his refutation of Ayer's weak verifiability principle. Ayer discussed the same sentence "All men are mortal" that serves as one of Berlin's examples of a sentence that should not turn out verifiable but it does — according to Ayer's original verification principle. It is unquestionable that Ayer does not give precise definitions along the lines of Carnap's observational and theoretical languages. However, if we read Ayer carefully, then his position may not be as simple (not to say as simplistic) as the criticisms of his work in [14] and [4] might suggest. After presenting the problem of universals, that is, their importance for the sciences and the impossibility of their conclusive verifiability, Ayer relaxes the requirement of establishing the truth or falsity of a statement as a sine qua non for meaningfulness. For the latter, he requires that a relevant observation could produce a truth value. Berlin sharply opposes the idea of using relevance in any determination of meaningfulness. He claims [4, p. 233] that "[r]elevance is not a precise logical category, and fantastic metaphysical systems may choose to claim that observation data are 'relevant' to their truth. . . . As a criterion for distinguishing sense from nonsense relevance plainly does not work: indeed to accept it is in effect to abrogate the principle of verification altogether."

In the 1930s, there was no widely known relevance logic; hence, neither Ayer nor Berlin could refer to a formal system of reasoning. The paradoxes of material implication, however, were widely known, and C. I. Lewis have made an attempt to remedy the *irrelevance* of 2-valued logic. His strict implication suffers from similar paradoxes, moreover, some of his systems (**S4** and **S5**) have materially paradoxical theorems too. In other words, he did not succeed in getting rid of the paradoxes of material implication, but he laid the foundations of the field of (formal) modal logics, which has been flourishing for more than a century.

Having formulated a liberal verifiability criterion, Ayer remarks [3, p. 39] that "[i]n contrast to the principle of conclusive verifiability, it clearly does not deny significance to general propositions or to propositions about the past." Berlin rephrases Ayer as follows: "given three propositions p, q, r, where r is conclusively verifiable in principle, then p is weakly verified, and therefore significant, if r follows from p and q, and does not follow from q alone." [4, p. 235] His concrete example, which incidentally, combines universality with past and future tense, goes like this: p is "All men are mortal," q is "Socrates is a man" and r is "Socrates will die." Berlin does not seem to have any problem with Socrates having been long dead or with the conclusion apparently pertaining to the future. He remarks [4, p. 234] that "[i]t may be noted that 'verifiable' seems here to have lost its sense of 'rendered true' or 'established beyond doubt' . . . ." This is in agreement with Ayer's insistence what the proper question to ask is, but seems to be a problem for Berlin. To get to the idea that meaningfulness cannot be the result of verification, he gives [4, p. 234] a curious example (of which he seems to think that it has the same logical form as the previous inference).

> This logical problem is bright green.
> I dislike all shades of green.
> Therefore, I dislike this problem.

The inference does not look like the previous inference to somebody who is familiar with 2-valued predicate logic. Perhaps, the most striking problem is that the second premise relates colors and the utterer of the sentence rather than objects of a certain color and the utterer of the sentence.[9] A better way to phrase the premise would be as "I dislike everything that is of a shade of green." Of course, the pronouns ("I" and "this") would require some explaining or disambiguation, because these expressions refer variably depending on context. Lastly, just as "is mortal" and "will die" were loosely taken to be the same above, "is bright green" and "is of a shade of green" seem to be identified. In the latter case, an unfortunate stumbling block to such an identification is that bright green is merely one of many shades of green.

If we take Ayer's formulation to be a *definition* with no restrictions on the types of premises that might be used, then it allows for a wide range of sentences to be labeled as weakly verifiable. The inference pattern

$$(\star) \qquad \{\, Pa, \forall x\,(Px \supset Qx) \,\} \vdash Qa$$

holds in 2-valued predicate logic, when P and Q are predicates, a is a name constant and $\forall$ and $\supset$ are the universal quantifier and the conditional connective. (We use "$\vdash$," which might suggest a *syntactic* consequence relation, but in 2-valued logic it's equipotent to a semantic consequence relation, hence, this bit of notation here does not have special syntactic significance. Similarly, "$\nvdash$" simply indicates that the consequence does not hold.) Then Ayer's requirement can be formally captured as

$$P_1 \wedge \cdots \wedge P_n \wedge Q \vdash O, \qquad \text{but} \qquad P_1 \wedge \cdots \wedge P_n \nvdash O,$$

where O is the only sentence that must be *experiential* or *observational*.[10]

If the Socrates example matches ($\star$), then "All man are mortal" is verified once Socrates has been observed to die. Of course, the other premise is also verifiable, but that may be less surprising, because if Socrates can be seen to die, then it is plausible that he can be seen to be a man too. If we massage the premises in Berlin's second example to fit ($\star$), then the added surprise (to a universal statement being verified by a single observation) is that the verification is done with the help of a premise that seems not to make much sense — unlike the universal statement and the observational sentence.

Carnap and Ayer put forward multiple caveats about universally quantified statements. Carnap, who appears to have been more formally inclined than Ayer, so to speak, often reduced quantified sentences to finite conjunctions or disjunctions in his writings by stipulating a finite domain for quantification. It seems to us that Berlin's counter example is not super interesting, because a sentence of the form $\forall x\, Ax$ is syntactically distinguishable from a singular statement, and could be treated separately.

2. A revised verifiability principle

Ten years and a world war after the publication of Ayer's book, a second edition was published in 1946. The main difference between the two editions is a substantial introduction, in which Ayer acknowledges that his verification principle was too liberal. His goal was — in the spirit of logical positivism — to free philosophy from metaphysical

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[9] It would seem that making a distinction between a color and an object (that has that color) is something that a philosopher should be careful about, especially, in view of extensive debates about color terms in pre-20th century German philosophy.

[10] We have ignored that Ayer might have meant a conjunction of observational sentences $O_1 \wedge O_2$ or $O_1 \wedge \cdots \wedge O_m$ in place of a single factual sentence O.

nonsense. He himself gives an example that does not involve a universal statement, but includes a conditional. Indeed, the inference pattern

$$(^{**}) \qquad \{S, S \supset O\} \vdash O$$

suffices to support the conclusion "This is white" from "The Absolute is lazy. If the Absolute is lazy, then this is white." According to Ayer, the role of verification is not to check truth or falsity, but to *assign meaning* to sentences. Thus, we are not required to look and see if the Absolute is (or is not) lazy. The observation of the object referred to by "this" or merely the possibility to make such a physical observation grants meaning to both premises. This is not acceptable for Ayer, because he considers the Absolute to be part of "a reality transcending the world of science and common sense" [3, p. 33]. Ayer admits that he overlooked another aspect of empirical statements, namely, that they are frequently *vague*. We will not deal with vagueness here; we simply set aside that issue.

To avoid the conclusion that all indicative statements have meaning, Ayer proposes a more subtle principle, which is to be applied to statements that are not analytic. All analytic statements are meaningful. We quote him [3, p. 13]:

> ... a statement is directly verifiable if it is either itself an observation-statement, or is such that in conjunction with one or more observation-statements it entails at least one observation-statement which is not deducible from these other premises alone; and I propose to say that a statement is indirectly verifiable if it satisfies the following conditions: first, that in conjunction with certain other premises it entails one or more directly verifiable statements which are not deducible from these other premises alone; and secondly, that these other premises do not include any statement that is not either analytic, or directly verifiable, or capable of being independently established as indirectly verifiable.

We reproduced the revised verifiability principle in its entirety, because of its importance. And now we give a definition. ($\bigwedge$ is a generalization of $\wedge$, which is applicable to finitely many sentences.)

Definition 2.1 (Verifiable sentences)**.** (1) Observational sentences are *directly verifiable.* (2) If $O_1, \ldots, O_n$ ($n \in \mathbb{N}$) and O' are observational sentences, and $\bigwedge_{i=1}^{n} O_i \wedge P \vdash O'$ whereas $\bigwedge_{i=1}^{n} O_i \nvdash O'$, then P is *directly verifiable.* (3) If $Q_1, \ldots, Q_n$ ($n \in \mathbb{N}$) are directly verifiable or analytic sentences, and Q' is a directly verifiable sentence, and $\bigwedge_{i=1}^{n} Q_i \wedge P \vdash Q'$ whereas $\bigwedge_{i=1}^{n} Q_i \nvdash Q'$, then P is *indirectly verifiable.* We denote the set of (directly or indirectly) verifiable sentences by $\mathbb{V}$.

The Definition 2.1 is slightly *more restrictive* than Ayer's, because in (3) we omitted "independently indirectly verifiable" statements. In Ayer's example (above), both premises could be claimed to be verifiable based on the inference (**), according to Ayer's original verifiability principle. We might conjecture that Ayer realized that using material implication could lead to a conditional like $S \supset O$ to be classified as indirectly verifiable, and then one more step would make S verifiable too. For example, without an "independence requirement," we could proceed as follows. Let O's be observational sentences that do not imply each other. $O_1 \wedge (S \wedge (S \supset O_2)) \vdash O_2$ renders $S \wedge (S \supset O_2)$ directly verifiable. But then $S \supset O_3$ is indirectly verifiable, because $(S \wedge (S \supset O_2)) \wedge (S \supset O_3) \vdash O_3$. Further, $(S \supset O_3) \wedge S \vdash O_3$ would make S indirectly verifiable, because $S \supset O_3$ is (already) indirectly verifiable. At least informally, it is clear that S played a role in establishing that $S \supset O_3$ is indirectly verifiable. Thus, this line of reasoning is excluded by the last clause in the last sentence in Ayer's verifiability principle. Our example also hints at the possibility that "dependence" might not be formalizable simply by appeal to a formula following from certain premises and not following from some of them separately. It is possible that for an indirectly verifiable sentence the "history" of its becoming independently verifiable would need to be defined and referenced in subsequent uses of the sentence in order to define independence.

In any case, Definition 2.1 is sufficient for Church's counter example that we state as Theorem 2.2; hence, we will not explicate "independently indirectly verifiable" further. We note though that our (3) gives the impression that indirectly verifiable sentences are just a step away from observational sentences, and this, perhaps, could lead to too few elements in the set $\mathbb{V}$. Something like this might have been the reason why Ayer allowed indirectly verifiable sentences — with a proviso — among the premises.

Alonzo Church in [10] gave a quick example, which not only alleviates any concerns about $\mathbb{V}$ being *too small* but shows that $\mathbb{V}$ is *too large*. It is reasonable to assume that Ayer, Church and other writers involved in verifiability debates were relying on 2-valued logic. There is no indication in the texts that anybody would have contemplated using intuitionistic logic for verification, which was the main alternative to 2-valued logic at the time.[11] Furthermore, the motivations for intuitionistic logic came from mathematics rather than the natural sciences. Thus, we formulate Church's counter example as the following theorem, given 2-valued logic.

Theorem 2.2. *Let S be an arbitrary sentence in the language. Let O_1, O_2 and O_3 be distinct observational sentences, which pairwise do not imply each other. Then S or $\neg S$ is* verifiable.

Proof. Let P be the sentence $(\neg O_1 \wedge O_3) \vee (\neg S \wedge O_2)$.
1. $O_1 \wedge P \vdash O_2$, by 2-valued logic, and $O_1 \nvdash O_2$ by our initial stipulation. Thus, P is directly verifiable.
2. $P \wedge S \vdash O_3$, by 2-valued logic. If $P \nvdash O_3$, then S is indirectly verifiable. If $P \vdash O_3$, then $\neg S \wedge O_2 \vdash O_3$, by 2-valued logic. Since $O_2 \nvdash O_3$, by assumption, $\neg S$ is directly verifiable. ∴

The existence of three independent observational sentences does not seem to be an onerous stipulation at all with respect to a scientific theory. Church was careful not to conclude from the direct verifiability of $\neg S$ the verifiability of S (i.e., $\neg\neg S$) itself. Indeed, it is quite plausible that negation does not need to preserve verifiability. It is not a new discovery that it is easy to verify an existential statement (if it is true), and it is easy to falsify a universal statement (if it is false), because in both cases a *single object* with the right properties or a *single observation* suffices. K. Popper's requirement that scientific laws must be capable of potential falsifiability is closely related to the asymmetry between the role of existential and universal sentences, and it was widely known among philosophers of science since the publication of his book in the 1930s. Ayer himself did not stipulate that verifiable sentences have properties such as that the

[11] Carnap *explicitly excluded*, for instance, in [7, 8], any intensional connectives; the implication of intuitionistic logic would have fallen into that category, because Gödel had proved, in 1933, that intuitionistic logic is not a many-valued logic with finitely many values.

negation of a verifiable sentence is verifiable. Of course, if *all* sentences are verifiable, then $\mathbb{V}$ is closed under the application of every logical connective and quantifier. However, this is a corollary of the theorem; it is not an assumption.

Some authors (e.g., Nidditch [16]) attempted to find a fault with Church's counter example. However, the counter example is *flawless* — if we assume that 2-valued (propositional) logic is the logic that is used. It is also easy to see that our slightly narrower definition of $\mathbb{V}$ does not interfere with omniverifiability. Thus, a fortiori, all sentences are verifiable according to Ayer's revised verifiability principle using 2-valued logic. Lastly, we may stress that Church's counter example is strongly *compelling*, because it does not involve any analytic sentences, quantifiers or conditionals.

3. A relevance logic for verifiability

Ayer as well as his critics broached the idea that experiences should be *relevant* to the sentences that are being verified. And they claimed in unison that relevance is vague or that logic cannot deal with relevance. Logic in the 1930s and even in the 1940s did not have a thoroughly worked out formal system that could have reflected pertinent connections between various types of statements. Today the situation is very different, because there are many systems of relevance logic to choose from; thus, we have to make a choice.

We shall use the *logic of relevant implication*, which is usually denoted by **R**. This logic, more precisely its implicational fragment, was introduced by Church in [11] and the motivation was closely related to the idea of relevance. Church was a mathematician and the inventor of the λ-calculus (cf. [9]), which he preferred in the form where functions do not have arguments that have no impact on the dependent variable's value. Such functions have a strong resemblance to deductions in which the set of premises is pared down to those assumptions that are *actually used* in the inference. **R** has other attractive features, which make it well suited for our purposes. The first full-blown relevance logic algebraized was **R** (see [12]), and the algebra of **R** can be characterized using properties familiar from other algebras. All (non-trivial) extensions of **R** (in its vocabulary) are subsystems of 2-valued logic under an obvious translation of the connectives. **R** has elegant axiomatizations, and 2-valued logic deviates from the relevant path by permitting as a theorem $A \supset (B \supset A)$. The latter is the *positive paradox*, one of the formulas that Lewis was trying (somewhat unsuccessfully) to exclude from the set of theorems at the beginning of the 20th century.

We give a concise exposition of the syntax and semantics of **R**, and then we scrutinize three arguments that intend to demonstrate problems with verifiability principles. We restrict our presentation to the propositional fragment, because that is sufficient to analyze the arguments. (See [1, 2, 13, 5] for detailed information on relevance logics.)

Definition 3.1. The language of **R** contains *propositional variables* $\{ p_i \}_{i \in \mathbb{N}}$ and *connectives* $\sim$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\circ$ (fusion), $\rightarrow$ (implication) and $\boldsymbol{t}$ (truth). Formulas are generated by the following CFG (context-free grammar) in which $\mathbb{P}$ rewrites to a p_i.

$$A := \mathbb{P} \mid \boldsymbol{t} \mid {\sim}A \mid (A \wedge A) \mid (A \vee A) \mid (A \circ A) \mid (A \rightarrow A)$$

The *axioms* and *rules* of **R** are (A1)–(A18) and (R1)–(R2).[12]

(A1) $A \to A$
(A2) $(A \to B) \to ((C \to A) \to (C \to B))$
(A3) $(A \to B) \to ((B \to C) \to (A \to C))$
(A4) $(A \to (B \to C)) \to (B \to (A \to C))$
(A5) $(A \to (A \to B)) \to (A \to B)$
(A6–7) $(A \wedge B) \to A, \qquad (A \wedge B) \to B$
(A8) $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$
(A9–10) $A \to (A \vee B), \qquad B \to (A \vee B)$
(A11) $((A \to B) \wedge (C \to B)) \to ((A \vee C) \to B)$
(A12) $(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$
(A13) $(A \to {\sim}B) \to (B \to {\sim}A)$
(A14) ${\sim}{\sim}A \to A$
(A15) $((A \circ B) \to C) \to (A \to (B \to C))$
(A16) $(A \to (B \to C)) \to ((A \circ B) \to C)$
(A17) $(\boldsymbol{t} \circ A) \to A$
(A18) $A \to (\boldsymbol{t} \circ A)$
(R1) $A \to B,\ A \ \Rightarrow\ B$
(R2) $A,\ B \ \Rightarrow\ A \wedge B$

A logic may be interpreted using a various approaches, and there are many publications on interpretations of relevance logics, in general, and specifically, on semantics of **R**; we refer to [13] and [2], where various semantics may be found together with an extensive bibliography.

In the possible-world semantics for modal logics, the worlds are thought to be snapshots of worlds. There are many "worlds" in the previous sentence, because the objects (or points) in a structure are called worlds, and informally they are explained as complete and consistent descriptions of *the* existing world and its alternatives. In the semantics of **R**, the objects are *situations*, which may not be complete, and they may contain a sentence and its negation. It is fair to say that situations are *more realistic descriptions* of the world than possible worlds, because despite our best efforts, we might have information that is neither complete nor consistent. Arguably, situations underpin a model of reasoning that is more faithful to practice than what the highly idealized possible worlds can carry.

Definition 3.2. A *structure* for **R** is a quadruple $\mathfrak{F} = \langle U, I, R, {}^* \rangle$, where the elements satisfy the conditions listed in (c1)–(c8).[13]

(c1) $U \neq \emptyset;\quad I \subseteq U;\quad I \neq \emptyset;\quad R \subseteq U \times U \times U;\quad {}^*\colon U \longrightarrow U;$
(c2) $\alpha \sqsubseteq \beta \ \Leftrightarrow\ \exists\iota\, R\iota\alpha\beta;\quad \alpha \sqsubseteq \alpha;\quad \iota \sqsubseteq \alpha \ \Rightarrow\ \alpha \in I;$
(c3) $(R\alpha\beta\gamma \,\&\, \alpha' \sqsubseteq \alpha \,\&\, \beta' \sqsubseteq \beta \,\&\, \gamma \sqsubseteq \gamma') \ \Rightarrow\ R\alpha'\beta'\gamma';$
(c4) $\exists\vartheta\,(R\alpha\beta\vartheta \,\&\, R\vartheta\gamma\delta) \ \Leftrightarrow\ \exists\vartheta\,(R\alpha\vartheta\delta \,\&\, R\beta\gamma\vartheta);$

[12] There are various ways to axiomatize **R**. We point to [1] and [2], where variations on this axiomatization as well as other proof systems for **R**, for its fragments and for further relevance logics may be found.

[13] We use lower-case Greek letters as variables for elements of U, in particular, ι's are elements of I. $\sqsubseteq$ is a defined binary relation. All the variables that are not quantified explicitly (by $\exists$), are implicitly universally quantified, that is, we take universal closures of the conditions. The symbols &, $\Rightarrow$ and $\Leftrightarrow$ stand for conjunction, implication and co-implication, respectively, in the metalanguage.

(c5) $\exists\vartheta\,(R\alpha\beta\vartheta \,\&\, R\vartheta\gamma\delta) \;\Rightarrow\; \exists\vartheta\,(R\alpha\gamma\vartheta \,\&\, R\vartheta\beta\delta)$;
(c6) $R\alpha\beta\gamma \;\Rightarrow\; \exists\vartheta\,(R\alpha\beta\vartheta \,\&\, R\vartheta\beta\gamma)$;
(c7) $\alpha^{**} = \alpha$;
(c8) $R\alpha\beta\gamma \;\Rightarrow\; R\alpha\gamma^*\beta^*$.

A *model* for **R** is a quintuple $\mathfrak{M} = \langle U, I, R, {}^*, v\rangle$, where $\mathfrak{F}$ is as above, and v satisfies (v1)–(v7).

(v1) $(\alpha \in v(p) \wedge \alpha \sqsubseteq \beta) \;\Rightarrow\; \beta \in v(p)$, where p is a propositional variable;
(v2) $v(\boldsymbol{t}) = I$;
(v3) $\alpha \in v(A \wedge B)$ iff $\alpha \in v(A)$ and $\alpha \in v(B)$;
(v4) $\alpha \in v(A \vee B)$ iff $\alpha \in v(A)$ or $\alpha \in v(B)$;
(v5) $\alpha \in v({\sim}A)$ iff $\alpha^* \notin v(A)$;
(v6) $\alpha \in v(A \circ B)$ iff $\exists\beta\,\exists\gamma\,(R\beta\gamma\alpha \,\&\, \beta \in v(A) \,\&\, \gamma \in v(B))$;
(v7) $\alpha \in v(A \to B)$ iff $\forall\beta\,\forall\gamma\,((R\alpha\beta\gamma \,\&\, \beta \in v(A)) \Rightarrow \gamma \in v(B))$.

Remark 3.3. We should highlight some features of the interpretation, which are absent in the semantics of modal logics and are of interest for our purposes. The set I may be thought to be the set of *logical situations*, and *validity* is defined with reference to this set. The interpretations of $\wedge$ and $\vee$ look fairly usual, because they are similar to the interpretation of these connectives in modal logic. On the other hand, $\sim$'s clause (v5) involves *, which shifts our attention from α to another situation α^* and its relationship to the formula in the scope of $\sim$. This is, obviously, different than the interpretation of $\neg$ in modal logics, though disregarding * altogether would return $\neg$ to us. Finally, $\circ$ has certain similarities to $\wedge$ as well as to $\Diamond$ (the modal operation called possibility), and $\to$ has certain similarities to intuitionistic implication as well as to $\Box$ (the modal operation called necessity).

3.1. **Ayer's example.** Let us start with the example that Ayer gave, that is, the inference pattern (**). This kind of inference is problematic for his original verifiability principle from 1936. It is well known that 2-valued propositional logic has *one theorem*, namely, $A \vee \neg A \vee B$, where B may be missing or it can be an arbitrary formula. The connective $\supset$ is not in the language of **R**, however, we might consider how $\supset$ could be emulated in **R**. In other words, having rewritten the second premise in (**) as $\neg S \vee O$, we are wondering what to do with $\neg$. As a matter of fact, for any formula in the language of **R**, $A \vee {\sim}A \vee B$ is a theorem. Thus, we could define $A \supset B$ as ${\sim}A \vee B$, and then ask if (**) is a correct inference. The answer is that, in general, (**) does not hold in **R**, it holds though when both premises are theorems of **R**. This is the famous result of R. K. Meyer and J. M. Dunn [15] about the admissibility of the rule γ in **E**, **R** and **T**. To be more precise, the result is that detachment is OK for material implication when the premises are logical truth. The other half, that is, that detachment is not OK for arbitrary premises may be shown similarly as we show below the failure of the relevant consequence relation in Church's example.

There is another way to look at (**). The formula $S \supset O$ is some sort of conditional formula, so we might think that a suitable way to transmogrify it into **R** is as $S \to O$. Of course, this looks like cheating, because thereby we have sneaked some *relevance* into the formula, figuratively speaking. Although relevance logics do not contain a marker that is attached to a formula or an inference to signal its relevance, an implication that is provable *does capture* the idea that the antecedent and the consequent are relevant to each other in the form of the variable sharing principle. To cut to the chase, $\{ S, S \to O \} \vdash O$ is a correct inference in **R**; it's simply the rule (R1), which is variably called detachment or implication elimination.

To summarize, Ayer's own example does or does not work against his original verifiability principle — depending on the formalization of the second premise as a conditional or as an implication. We think that it is better to keep formalizing the sentence "If the Absolute is lazy, then this is white" as a material implication (i.e., $S \supset O$) rather than as a relevant implication (i.e., $S \to O$).

3.2. **Church's example.** Theorem 2.2 contains the reasoning that Church presented in [10] to show that each sentence or its negation is verifiable. We have already mentioned that the example is very powerful, because it assumes only three observational sentences, and of course, 2-valued logic. For instance, all tautologies and contradictions slip into the set of verifiable sentences, despite that — thinking with Wittgenstein — they do not say anything about the world.[14]

The formula P at the center of Church's counter example is formulated using so-called Boolean connectives, which can be easily mapped into the language of **R**. If we replace $\neg$ with $\sim$, then we get a formula $(\sim O_1 \wedge O_3) \vee (\sim S \wedge O_2)$ that we denote as P'. Now the question is whether steps 1 and 2 are valid inference steps in **R**. If we look at the proof of Theorem 2.2, then we can easily determine how to get from $O_1 \wedge P'$ to O_2. For instance, we might consider the following series of steps. $O_1 \wedge ((\sim O_1 \wedge O_3) \vee (\sim S \wedge O_2))$ is equivalent to $(O_1 \wedge \sim O_1 \wedge O_3) \vee (O_1 \wedge \sim S \wedge O_2)$, by the distribution of $\wedge$ over $\vee$ into $\vee$. We have dropped some parentheses (because of the associativity of $\wedge$), which should help us to see that the first parenthesized part of this formula contains a contradiction. In 2-valued logic, $(A \wedge \neg A) \vee B$ reduces to B. Then (originally) O_2 follows, because $O_1 \wedge \neg S \wedge O_2 \vdash O_2$. What we said about situations and their more tolerant nature should suggest that the reduction step due to a contradiction is at least *suspicious* in **R**. Indeed, neither step 1 nor step 2 are valid in **R**. The properties of conjunction and disjunction are similar in 2-valued logic and in **R**, however, the way they interact with negation differs.

The two steps in Church's argument and the detachment from a material implication in Ayer's example amount to uses of the *disjunctive syllogism* rule (together with some other OK steps). In the relevance logic literature, this rule is often referred to as the γ rule, because this was the third rule in W. Ackermann's Π' calculus, and it was dropped by A. R. Anderson and N. D. Belnap in their formulation of **E**, the logic of entailment. (Cf. [2] for some historical remarks.) It is also well known from [13, p. 151] how the use of the disjunctive syllogism in Lewis's "proof" leads to an arbitrary formula (as a conclusion) from a contradiction (as a premise). To deal with three steps at once, we show that disjunctive syllogism is not a correct inference step using two arbitrary formulas A and B. That is, we show that $\{ A, \sim A \vee B \} \nvDash B$.[15]

[14] We quote Wittgenstein [18, 4.462]: "Tautology and contradiction are not pictures of reality. They present no possible state of affairs."

[15] The logic **R** is sound and complete for the semantics that we presented; hence, we could use $\nvdash$ in the claim. However, we give a semantic proof now and for this reason we use $\nvDash$, for harmony. We wish to emphasize that the failure of disjunctive syllogism is well known (see e.g., [1, §25.1] and [13, §1.8]); we simply sketch a little proof here specifically for **R** using relational semantics.

Lemma 3.4. *Let us consider the model* $\mathfrak{M}_\gamma = \langle U, I, R, {}^*, v\rangle$, *with the following components.*

(1) $U = \{\, \iota, \alpha, \beta \,\}$ *(there are three situations).*
(2) $I = \{\, \iota \,\}$ *(i.e.,* ι *is the logical situation).*
(3) $R = \{\, \langle x, x, x\rangle, \langle \iota, x, x\rangle, \langle x, \iota, x\rangle, \langle \alpha, \beta, \beta\rangle, \langle \beta, \alpha, \beta\rangle, \langle \beta, \alpha, \alpha\rangle, \langle \alpha, \beta, \alpha\rangle, \langle \alpha, \beta, \iota\rangle,$ $\langle \beta, \alpha, \iota\rangle \,\}$, *where a triple with* x *is instantiated with each element of* U *for* x.
(4) $\iota^* = \iota,\ \alpha^* = \beta,\ \beta^* = \alpha$.
(5) $v(A) = \{\, \alpha \,\}$ *and* $v(B) = \{\, \beta \,\}$.

Then, $\alpha \in v(A \wedge (\sim A \vee B))$, *but* $\alpha \notin v(B)$.

Proof. The proof is quite straightforward. A tedious bit is to check that R satisfies the required conditions, the details of which we omit. But we show that α is and is not in the considered propositions as desired. $\alpha \in v(A)$ and $\alpha^* \notin v(A)$, by definition. Then, $\alpha \in v(\sim A)$, hence, $\alpha \in v(\sim A \vee B)$. Thus, indeed, $\alpha \in v(A \wedge (\sim A \vee B))$. But $\alpha \notin v(B)$ by the definition of v. ⁖

We think that it is remarkable that Church's counter example is defused — as elegantly as his argument was in 2-valued logic — by progressing to a *more subtle logic* to which Church [11] himself contributed significantly by introducing the implicational fragment of the logic.

3.3. **Yi's example.** Finally, we analyze yet another counter example to a verifiability principle. We start with a caveat, namely, the verifiability principle in question from [19] is significantly different from Ayer's verification principles. For example, $\mathbb{V}$ is assumed to be closed under $\neg$, $\vee$ and $\exists$, and the underlying logic differs from 2-valued logic. Yi in [20] gave a counterexample, in which he uses "compact logic." Yi seems to have made some changes to Wright's compact logic from [19], but whether he did or did not, the logic he uses is not stronger than 2-valued logic. Our goal is to see if Yi's argument would refute Wright's verification principle if the logic is changed to **R**, that is, we intend to find out whether all sentences turn out to be verifiable. (We do not present compact logic here, because we will not deal with S-compact inferences at all.) We first outline Yi's reasoning. This will clarify the structure of his argument including the inference steps taken (which would be correct, for example, in 2-valued logic) and the assumptions used (which come from [19], and some of them might be doubted in some empirical sciences). We use symbols from 2-valued logic, because Wright in [19] seems to define S-compact inference bouncing it off from the classical consequence relation. We also hope that the familiar notation will help a reader to follow the steps.

As before, S is an arbitrary sentence and O is an observational sentence. O is verifiable, hence, so is $\neg O$. O is equivalent to $(S \supset O) \wedge (\neg S \supset O)$, hence the latter is verifiable, and by $S \supset O$ being a consequence, it is also verifiable. Similar steps lead from $\neg O$ to $(S \supset \neg O) \wedge (\neg S \supset \neg O)$, and then to $S \supset \neg O$ and to the claim that $S \supset \neg O$ is verifiable. By the closure properties of $\mathbb{V}$ in [19], then, $\neg(S \supset O) \in \mathbb{V}$, $\neg(S \supset \neg O) \in \mathbb{V}$, hence, $\neg(S \supset O) \vee \neg(S \supset \neg O) \in \mathbb{V}$ too. However, the last formula is equivalent to S, therefore, S is verifiable.

We have argued, inter alia, that closure of the set of verifiable sentences under negation is a doubtful assumption. Obviously, the construction and Yi's argument heavily rely on this assumption and on several other closure properties. However, let us grant those assumptions for the time being. In order to consider the argument in **R**, we have to decide which connectives correspond to which symbols. It seems that we can take $\wedge$ and $\vee$ to be conjunction and disjunction, respectively. Since Yi uses compact entailment, we shall match $\supset$ with relevant implication and we take the negation $\neg$ to be the negation $\sim$. In the argument, there is no application of the γ rule. However, a philosopher looking out for (ir)relevance surely notices how S and $\neg S$ first get tied up with O and $\neg O$, and then, the O's drop out.

The very first steps of the argument fail in relevance logic, because C is not equivalent to $(S \to C) \wedge ({\sim}S \to C)$ whether C is O or ${\sim}O$. $(S \to C) \wedge ({\sim}S \to C)$ is equivalent to $(S \vee {\sim}S) \to C$ and $S \vee {\sim}S$ is a theorem of **R**. However, $C \to ((D \vee {\sim}D) \to C)$ is not a theorem of **R**. Using our example from §3.2 (Lemma 3.4) and taking D to be B, and C to be $A \vee B$, we have that $\alpha \in \nu(C)$ but $\alpha \notin \nu((D \vee {\sim}D) \to C)$. It is easy to see that $\nu(D \vee {\sim}D) = \{\,\beta\,\}$ and $\nu(A \vee B) = \{\,\alpha, \beta\,\}$. There are five three-tuples that are of the form $\langle \_\,, \beta, \_\, \rangle$ and belong to R. Three of them have α in the first place. Using (v7) from Definition 3.2, we see that $R\alpha\beta\iota$ and $\beta \in \nu(B \vee {\sim}B)$, but $\iota \notin \nu(A \vee B)$; hence, $\alpha \notin \nu((D \vee {\sim}D) \to C)$. We have already noted that $S \vee {\sim}S$ is a theorem. However, in **R**, not all theorems are "equal" — or more precisely, some theorems do not imply some other theorems. $C \to (\boldsymbol{t} \to C)$ is easily seen to be a theorem, because $C \to C$ and $\boldsymbol{t}$ are theorems and (A4) is an axiom. But $(B \vee {\sim}B) \to \boldsymbol{t}$, in general, is *not a theorem* of **R**. If we were to compress the first two steps into $O \to (S \to O)$, and $S \to O \in \mathbb{V}$, because $O \in \mathbb{V}$, then the problem would become strikingly obvious. Namely, $C \to (E \to C)$ is not only not a theorem of **R** (for arbitrary C and E), but it's an anathema in relevance logic, exactly, because E may be irrelevant to C.

Similarly, ${\sim}(S \to O) \vee {\sim}(S \to {\sim}O)$ is not equivalent to S. Once again, dually to the previous case, ${\sim}(S \to O) \vee {\sim}(S \to {\sim}O)$ is equivalent to ${\sim}(S \to (O \wedge {\sim}O))$. However, the next step, which claims equivalence with S, fails. This time we will argue in a manner that is close to syntactic-style reasoning; we appeal to known logical equivalences of **R** together with the relevant character of the logic. Let's consider an instance of the entailment ${\sim}(S \to (O \wedge {\sim}O)) \to S$ with distinct propositional variables p and q. ${\sim}(p \to (q \wedge {\sim}q)) \to p$ transforms in a few steps into $(q \vee {\sim}q) \to (p \to p)$. A cascading series of logical equivalences such as ${\sim}(p \to (q \wedge {\sim}q)) \to p \Leftrightarrow {\sim}p \to (p \to (q \wedge {\sim}q)) \Leftrightarrow {\sim}p \to ({\sim}(q \wedge {\sim}q) \to {\sim}p) \Leftrightarrow {\sim}(q \wedge {\sim}q) \to ({\sim}p \to {\sim}p) \Leftrightarrow (q \vee {\sim}q) \to (p \to p)$ shows this. The last formula is an implication without any variable being shared by the antecedent and the consequent. Thus, the formula is not a theorem of **R**.

This brief section did not scrutinize Wright's formulation of the verifiability principle or his compact logic. Either of those tasks would require a paper in itself. Also, Yi's modifications to S-compact entailments would need to be separately analyzed. We restricted our attention to the steps in the counter example using a reasonable mapping of connectives into the language of **R** and we showed that the use a relevance logic would block some of the steps in the derivation of the counter example.

4. Conclusions

We have considered ideas, concerns and some formulations of verifiability principles. It seems to us that some of the early attacks on verifiability (in the 1930s) were somewhat misguided. We have found that Church's counter example (showing that Ayer's revised verifiability principle is way too permissive) is on the button, if we assume that Ayer, Church and others were taking 2-valued logic for granted. However, we noted Ayer's appeal to relevance, and showed that the use of a relevance logic, namely **R**, hampers the counter examples to Ayer's revised verifiability principle — essentially, because a contradiction does not imply an arbitrary formula. We also considered a relevant version of Yi's argument against Wright's verifiability principle, and showed that using **R** precludes reaching the conclusion in that counter example — essentially, because theorems, in general, are not logically equivalent to each other in **R**.

We think that it would be a promising project to rework scientific theories of various kinds in the framework of relevance logic. Needless to say that such an endeavor is well beyond the scope of this paper.

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