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SOLVING HIGHER ORDER DIFFERENTIAL EQUATIONS: AN INTEGRATING FACTOR APPROACH AND GENERALIZED SOLUTIONS TO SELF-ADJOINT DIFFERENTIAL EQUATIONS

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Solving Higher Order Differential Equations: An Integrating Factor Approach and Generalized Solutions to Self-Adjoint Differential Equations

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Abstract

In this work we present an integrating factor approach for solving certain higher order differential equations (DE's) and a generalized method for solving Self-Adjoint Differential Equations.

1 Introduction

Modeling by using differential equations cuts across various fields [8]. The widespread applications of differential equations plays into its continuous relevance in the research community, yielding varying approaches to solving them analytically or numerically. Many physical phenomena are modeled by using differential equations. In Physics, the Legendre differential equation arises in the study of the Hydrogen atom wave functions and angular momentum in single-particle quantum mechanics [[4] [3]]. In the health sciences we have seen many differential equations being used for modeling complex systems ranging from modeling of evasive cancers [[6] [9]] to the spread of diseases [[5] [10]].

There are analytic methods available as means of solving some higher order differential equations, see [2] and the references therein and the Riccati-type approach [3]. For instance in [4] the authors used a change of variable approach to solve some self-adjoint differential equations. In addition, for higher order DE's where analytic solutions may be unattainable, numer-ical methods have been used with continued focus on error minimization [12]. In [7] the authors reformulated the spline approach which increases the accuracy of the second order approximation, thereby improving the en-tire accuracy of the numerical approximation to a third-order accuracy for an nonuniform mesh and a forth-order accuracy for a uniform mesh.

Many mechanical systems describing the motion of particles or bodies are modeled by second-order differential equations. However, some physical systems exhibit much more complex behaviors and thus may require the specification of third or higher order derivatives. In [3], the authors generalized the solution to second order and third order self-adjoint equations in addition to solving higher order Riccatti-type DE's. This paper extends the work done in [[2] [3]] by generalizing the solutions of higher-order self-adjoint DE's as well as using an integrating-factor approach to solve other special higher-order DE's.

By generating a 4th and 5th order self-adjoint ODE, we observe the solution of the 2nd and 4th order equations as well as the third and 5th order in order to draw a generalized solution for self-adjoint DE's of even and odd order respectively. Since the solutions to the second and third order have been studied in [3], we only reference them in this work. For applications and examples of 4th order ODEs see for example [17], [18]

In this paper we propose an approach for solving higher order differential equations by using solutions from its associated lower order equations. We observe that the integral of the associated lower order DE is a solution to its higher order associated DE. Thus we are able to derive solutions to higher order DE's by beginning with an associated lower order DE.

The paper is organized as follows: Section 2 uses an arbitrary ODE to lay out the discussed solution approach. In section 3 we present Matlab solutions to the derived higher order ODE as a means of comparison with the solution derived from our approach. In section 4 we present examples using the integrating-factor approach. Section 5 generalizes the solution to self-adjoint DE's by focusing on patterns drawn from even and odd orders. Finally In Section 6 we present the conclusions.

2 Integrating-Factor Approach

2.1 Second Order Ordinary Differential Equations

Let y satisfy:

$$y'' + p(x)y' + q(x)y = 0. \tag{1}$$

If we set b $b = \frac{-y'}{y}$ then b satisfies $b' = b^2 - p(x)b + q(x)$ and if

$$y'' + p(x)y' + q(x)y = f(x) \tag{2}$$

Then to find a particular solution for this equation we use results from [2] and [3] i.e,

$$[\mu[y' + by]]' = \mu f(x). \tag{3}$$

Which leads to $\frac{\mu'(x)}{\mu} = p(x) - b(x)$ and $(ub)' = \mu q$. thus $\frac{u'}{u} = p - b \rightarrow ln\mu = \int pdx + \int \frac{y'}{y}$ and $b = \frac{-y'}{y}$ so $ln\mu = \int pdx + \int \frac{y'}{y}dx = \int pdx + lny$ with $\mu = e^{\int pdx}y$. Substituting μ into $(ub)' = \mu q$ we get

$$\left(e^{\int pdx}y(\frac{-y'}{y})\right)' = \left(e^{\int pdx}(-y')\right)' = e^{\int pdx}yq \tag{4}$$

Thus:

$$e^{\int pdx}(-y'') - y'e^{\int pdx}p = e^{\int pdx}yq \tag{5}$$

and

$$e^{\int pdx}\left(y'' + py' + qy\right) = 0 \tag{6}$$

As $e^{\int pdx}$ is always positive, then $(y'' + by' + qy) = 0$
and we observe that equation 4 in terms of b yields:

$$\left[e^{\int pdx}(pb - b^2 + b')\right] = e^{\int pdx}q \tag{7}$$

then
$b' = q + b^2 - pb$
that is a first order Riccati equation.

So if $u = e^{\int pdx} y$ then replacing in equation 4 we obtain that y satisfies $y'' + py' + qy = 0$, so if b satisfies $b' = b^2 - pb + q$ then if we set $b = \frac{-y'}{y}$ then y satisfies $y'' + p(x)y' + qy = 0$.

On the other hand since $\frac{u'}{u} = p - b \implies b = p - \frac{u'}{u}$, which when substituted in 4 leads to $(ub)' = uq \implies (u(p - \frac{u'}{u}))' = uq$ and we get $\left(u\left(\frac{pu-u'}{u}\right)\right)' = uq = (pu - u')' = uq$ which implies:

$$pu' + p'u - u'' = uq$$

$$u'' - p'u - pu' + uq = 0 \tag{8}$$

with $u = e^{\int pdx} y$.

Where y is a solution of $y'' + p(x)y' + q(x)y = 0$ and then u satisfies $u'' - pu' + (q - p')u = 0$.

As $u = e^{\int pdx} y$ with y a solution of

$$y'' + p(x)y' + q(x)y = 0 \tag{9}$$

Then knowing a solution of 9 we obtain a solution of u and vice-versa. Knowing a solution of u that satisfies equation 8, we can have a solution for y given by equation 9 by the relation $u = e^{\int pdx} y \implies y = ue^{-\int pdx}$.

We now prove a result following the theory from [3] which states that:

Theorem 2.1. *Consider a general second order ODE,*

$$y'' + p(x)y' + q(x)y = F(x), \tag{10}$$

where $p(x), q(x)$ and $F(x)$ are continuous functions.

if u is the integrating factor and u satisfies

$$\frac{u'}{u} = p - b \tag{11}$$

with

$$b = \frac{-y_h'}{y_h} \tag{12}$$

where y_h satisfies

$$y_h'' + p(x)y_h' + q(x)y_h = 0, \tag{13}$$

Then if u satisfies

$$u'' - p(x)u' + (q - p')u = g(x) \tag{14}$$

and if y is a solution of equation 10 then $u = ye^{\int pdx}$ is a solution of 14 with $g(x) = e^{\int pdx}F(x)$

Proof. If y is a solution of of 10 then

$$y'' + p(x)y' + q(x)y = F(x) \tag{15}$$

and as $u = y_h e^{\int pdx}$ from 11 and 12, integrating 11 $lnu = \int pdx + \int \frac{y_h'}{y_h}dx = \int pdx + lny_h \rightarrow u = e^{\int pdx}y_h$, so if $y = \mu e^{-\int pdx}$ then computing and replacing u, u' and u'' into 14 we get:

$$e^{\int pdx}[y'' + 2py' + (p^2 + p')y] - P(x)[y' + py] + [q - p']y = g(x). \tag{16}$$

Which simplifies to

$$e^{\int pdx}[y'' + py' + qy] = g(x). \tag{17}$$

Now if $g(x) = 0$ then

$$u'' - pu' + (q - p')u = 0$$

as y satisfies $y'' + py' + qy = 0$. If $g(x) \neq 0$, then from the previous equation: $e^{\int pdx}[y'' + py' + qy] = g(x)$ we have that: $y'' + py' + qy = g(x)e^{-\int pdx}$. So if $F(x) = g(x)e^{-\int pdx}$ then u satisfies 14 as required.

On the other hand, if u satisfies 14, then $y = ue^{-\int pdx}$ satisfies 10 with $F(x) = g(x)e^{-\int pdx}$. To see this we replace in 10 $y = ue^{-\int pdx}$ to get

$$\left(ue^{-\int pdx}\right)'' + p(x)(ue^{-\int pdx})' + que^{-\int pdx} = F(x) \tag{18}$$

Expanding and simplifying equation 18 we get

$$u^{''}e^{-\int pdx} - pu^{'}e^{-\int pdx} + (q - p')ue^{-\int pdx} = F(x) \tag{19}$$

So if $y = \mu e^{-\int pdx}$ and u satisfies 14 then $y = ue^{-\int pdx}$ satisfies

$$[u'' - pu' + (q - p')u]e^{-\int pdx} = F(x) \tag{20}$$

So taking $F(x)e^{\int pdx} = g(x)$, we see that if $y = ue^{-\int pdx}$ then if equation 14 holds, it follows that equation 10 holds with $g(x) = F(x)e^{\int pdx}$. Which ends the proof. □

Now given $u = e^{\int pdx} y$ where y satisfies the homogeneous part of equation 10, then we have that $u = e^{\int pdx} y$ satisfies the homogeneous part of 14. To find the particular solution for equation 10 we use results from [3] which results in the solution:

$$e^{\int pdx} y_h (y' - \frac{y_h'}{y_h} y) = \int e^{\int pdx} y_h F(x) + C \tag{21}$$

And equation 21 can be rewritten as:

$$\left(y' - \frac{y_h'}{y_h} y\right) = \frac{1}{y_h} e^{-\int pdx} \int y_h e^{\int pdx} F(x) dx + \frac{c}{y_h} e^{-\int pdx} \tag{22}$$

Using the first integrating factor theory on equation 22 results in the expression below after some simplifications:

$$\frac{y}{y_h} = \int \frac{1}{{y_h}^2} e^{-\int pdx} \int y_h e^{\int pdx} F(x) d(x) + c \int \frac{e^{-\int pdx}}{y_h^2} dx + K \tag{23}$$

From equation 23 we get the solution:

$$y = y_h \int \frac{1}{y_h^2} e^{-\int pdx} \int y_h e^{\int pdx} F(x) d(x) + c y_h \int \frac{e^{-\int pdx}}{y_h^2} dx + K y_h \tag{24}$$

Now, if we have $h(x) = e^{\int pdx} F(x)$, then it is possible to prove that a solution of the differential equation:

$$\bar{y''} - p\bar{y'} + (q - p)\bar{y} = h(x), \tag{25}$$

is $\bar{y} = e^{\int pdx} y$ where y satisfies equation 10

And the solution to equation 25 is :

$$\bar{y} = e^{\int pdx} y_h \int \frac{e^{-\int pdx}}{y_h^2} \int y_h h(x) dx + \bar{c} e^{\int pdx} y_h \int \frac{e^{-\int pdx}}{y_h^2} + \bar{K} e^{\int pdx} y_h \tag{26}$$

If we replace $h(x) = F(x) e^{\int pdx}$ in equation 26 and simplify we arrive at the result:

$$\bar{y} = e^{\int pdx} [y_h \int \frac{e^{-\int} pdx}{y_h^2} \int y_h F(x) e^{\int pdx} + \bar{c} y_h \int \frac{e^{-\int pdx}}{y_h^2} dx + \bar{K} y_h] \tag{27}$$

So, if $h(x) = e^{\int pdx} F(x)$, then, $\bar{y} = e^{\int pdx} y$ where y satisfies equation 10.

To find the particular solution for equation 25 we again utilize results from [3] by setting $\bar{p} \to -p$ so that we have

$$\frac{\bar{u'}}{\bar{u}} = \bar{p} - \bar{b} = -p + \frac{\bar{y'_h}}{\bar{y_h}}$$

where $\bar{u}$ satisfies:

$$\bar{u}'' + p\bar{u}' + q\bar{u} = 0 \tag{28}$$

From which we get $\bar{u} = e^{-\int p dx} e^{ln \bar{y_h}}$. Recall that $y_h = e^{-\int p dx} u$ for y_h satisfying $y''_h + py'_h + qy_h = 0$. So a solution for equation 25 following results from [3] will be:

$$[e^{-\int p dx} \bar{y_h} (\bar{y}' - \frac{\bar{y'_h}}{\bar{y_h}} \bar{y})]' = e^{-\int p dx} \bar{y_h} h(x) \tag{29}$$

Equation 29 simplifies to:

$$\left(\bar{y}' - \frac{\bar{y'_h}}{\bar{y_h}} y \right) = \frac{1}{\bar{y_h}} e^{\int p dx} \int e^{-\int p dx} \bar{y_h} h(x) dx + \bar{c} \frac{e^{\int p dx}}{\bar{y_h}} \tag{30}$$

Applying the first integrating factor theory on equation 30 we get

$$\left(\bar{y_h}^{-1} \bar{y} \right)' = \frac{1}{\bar{y_h}^2} e^{\int p dx} \int e^{-\int p dx} \bar{y_h} h(x) dx + \bar{c} \frac{e^{\int p dx}}{\bar{y_h}^2}.$$

From which we arrive to the solution:

$$\bar{y} = \bar{y_h} \int \frac{e^{\int p dx}}{\bar{y_h}^2} \int e^{-\int p dx} \bar{y_h} h(x) dx + \bar{c} \bar{y_h} \int \frac{e^{\int p dx}}{\bar{y_h}^2} dx + \bar{K} \bar{y_h} \tag{31}$$

So as $u = e^{\int p dx} y_h$, then $u^2 = y_h^2 e^{2\int p dx} = \bar{y_h^2}$. Since u satisfies the homogeneous part of equation 14 and $\bar{y_h}$ is a homogeneous solution of equation 25 it follows that $u = \bar{y_h}$ which then leads to the general solution:

$$\bar{y} = e^{\int p dx} y_h \int \frac{e^{\int p dx}}{\bar{y_h}^2 e^{2\int p dx}} \int e^{-\int p dx} e^{\int p dx} \bar{y_h} h(x) dx + \bar{c} \bar{y_h} e^{\int p dx} \int \frac{e^{\int p dx}}{\bar{y_h}^2 e^{2\int p dx}} dx + \bar{K} \bar{y_h} e^{\int p dx} \tag{32}$$

We recall that $u = e^{\int p dx} y_h$, then we have a solution for y and $\bar{y}$. Substituting $h(x) = f(x) e^{\int p dx}$ into the solution of $\bar{y}$ we get:

$$\bar{y} = e^{\int pdx}\left[y_h \int \frac{e^{-\int pdx}}{y_h^2}\int y_h f(x) e^{\int pdx} dx + \bar{c} y_h \int \frac{e^{-\int pdx}}{y_h^2} + \bar{K} y_h\right] \tag{33}$$

Thus setting $h(x) = F(x)e^{\int pdx}$ we get $\bar{y} = e^{\int pdx} y$, where y and $\bar{y}$ satisfy equations 10 and 25 respectively.

Example 2.2. *Consider the equation below:*

$$y'' + 5xy' = 1 \tag{34}$$

The general solution for this equation is:

$$y = \int e^{\frac{-5}{2}x^2} \int e^{\frac{5}{2}x^2} dx + c \int e^{\frac{5}{2}x^2} + K.$$

For $\bar{y} = u$ with u as the integrating factor, from equation 34 $p = 5x$ and $q = 0$. Then from equation 14 and equation 25 u and $\bar{y}$ are solutions to the equation. Now, if $u = e^{\int pdx} y$ then, we have that y is the homogeneous solution of 34. Then substituting p into 27 we get:

$$u'' - 5xu' - 5u = 0.$$

Then with the relation between u and y stated above, we get the solution

$$y = c \int e^{\frac{5x^2}{2}} dx$$

and $y = K$ with K and c constants.
We have that $u = e^{\frac{5}{2}x^2} K$ is one possible u and the other possibility for u is $u = e^{\frac{5}{2}x^2} \int e^{\frac{-5}{2}x^2} dx$.

Therefore if $y = \int e^{\frac{-5}{2}x^2} dx$ satisfies the equation: $y'' + 5xy' = 0$ then $u = e^{\frac{5}{2}x^2} \int e^{\frac{-5}{2}x^2} dx$ satisfies $u'' - 5xu' - 5u = 0$.

Furthermore, if $h(x) = F(x)e^{\int pdx}$ then from equation 34 we get $h(x) = 1 * e^{\frac{5x^2}{2}} = e^{\frac{5x^2}{2}}$. Hence the solution for $\bar{y}'' - 5x\bar{y} - 5\bar{y} = h(x) = e^{\frac{5x^2}{2}}$ is $\bar{y} = e^{\int pdx} y = e^{\frac{5x^2}{2}} [\int e^{\frac{-5x^2}{2}} \int e^{\frac{5x^2}{2}} dx + c \int e^{\frac{-5x^2}{2}} dx + K]$. So $\bar{y} = e^{\frac{5x^2}{2}} [\int e^{\frac{-5x^2}{2}} \int e^{\frac{5x^2}{2}} dx + c \int e^{\frac{-5x^2}{2}} dx + K]$. Therefore we can go back and forth with the solutions y and $\bar{y}$.

2.2 Third Order Ordinary Differential Equation

Consider the third order differential equation below:

$$y''' + P(x)y'' + Q(x)y' + R(x)y = 0 \tag{35}$$

Then the solutions to equation 35 are related to the solutions to the equation:

$$y'' + by' + ay = 0 \tag{36}$$

where b satisfies the relation $\frac{u'}{u} = P - b$, and u is the integrating factor of equation 35 which satisfies the equation:

$$u''' - Pu' + (Q - 2P')u' + (Q' - P'' - R)u = 0 \tag{37}$$

As shown in [3] $a = Q - b' - b(P - b)$ and $b = \frac{-\bar{y}'}{\bar{y}}$ where $\bar{y}$ satisfies:

$$\bar{y}''' + 2P\bar{y}'' + (Q + P' + P^2)\bar{y}' + (Q' - R + QP)\bar{y} = 0 \tag{38}$$

So as shown in [3] we can apply the integrating factor technique on 35 that leads to the relation $u = e^{\int Pdx}\bar{y}$.

We recall that if

$$\bar{y}''' + 2P\bar{y}'' + (Q + P' + P^2)\bar{y}' + (Q' - R + QP)\bar{y} = 0$$

then,

$$(\mu(y'' + by' + ay))' = \mu 0 = 0$$

for details see [3].

We conclude that:

$$\mu(x) = e^{\int Pdx}\bar{y}$$

where $\bar{y}$ is a solution of 38.

Therefore

$$[e^{\int Pdx}\bar{y}(y'' + by' + ay)]' = 0$$

and putting everything in terms of b we get:

$$\mu(x) = e^{\int Pdx}e^{-\int bdx}$$

and using that (see [3]) $a = Q - b' - b(P - b)$ we obtain:

$$[e^{\int Pdx}e^{-\int bdx}(y'' + by' + ay)]' = 0$$

hence,

$$[e^{\int Pdx}e^{-\int bdx}(y'' + by' + ay)] = C$$

and we have that:

$$y'' + by' + ay = C(e^{-\int Pdx}e^{\int bdx})$$

and using the theory in [3] we obtain:

$$(\bar{\mu}(y' + \bar{b}y))' = \bar{\mu}Ce^{-\int Pdx}e^{\int bdx}$$

where $\bar{\mu}$ verifies

$$\frac{\bar{\mu}'}{\bar{\mu}} = b - \bar{b}$$

where $\bar{b} = -\frac{y_h'}{y_h}$ and y_h satisfies the equation:

$$y'' + by' + ay = 0$$

and replacing $\bar{\mu}$ and $\bar{b}$ we have:

$$\bar{\mu} = e^{\int bdx}y_h.$$

Substituting $\bar{\mu} = e^{\int bdx}y_h$ into the integrating factor expression we get $(e^{\int bdx}y_h(y' + \bar{b}y))' = e^{\int bdx}y_h ce^{-\int Pdx}e^{\int bdx}$ and substituting $\bar{b} = \frac{-y_h'}{y_h}$ into the equation above results in:

$$e^{\int bdx}y_h(y' + \frac{-y_h'}{y}y))' = e^{2\int bdx}y_h ce^{-\int Pdx}.$$

The equation above simplifies to:

$$(y' + \frac{-y_h'}{y}y) = \frac{e^{-\int bdx}}{y_h}c\int \frac{e^{2\int bdx}y_h}{e^{\int Pdx}}dx + \frac{K}{y_h}e^{-\int bdx}.$$

Applying the integrating factor of first degree yields:

$$(\bar{u}y)' = \bar{u}[\frac{e^{-\int bdx}}{y_h}c\int \frac{e^{2\int bdx}y_h}{e^{\int Pdx}}dx + \frac{K}{y_h}e^{-\int bdx}]$$

where $\bar{u} = y_h^{-1}$.

Which yields,

$$\frac{y}{y_h} = \int \frac{e^{-\int bdx}}{y_h^2} \int \frac{e^{2\int bdx} c y_h}{e^{\int Pdx}} dx + \int \frac{K e^{-\int bdx}}{y_h^2} dx + l$$

$\Longrightarrow$

$$y = y_h \int \frac{e^{-\int bdx}}{y_h^2} \int \frac{e^{2\int bdx} c y_h}{e^{\int Pdx}} dx + y_h \int \frac{K e^{-\int bdx}}{y_h^2} dx + l y_h$$

Therefore we have that:

$$K y_h \int \frac{e^{-\int bdx}}{y_h^2} dx + L y_h$$

is the general solution for $y'' + by' + ay = 0$. So then we conclude that the solutions to equation 35 are related to the the solutions of $y'' + by' + ay = 0$ where $b = \frac{-\bar{y}'}{\bar{y}}$ with $\bar{y}$ satisfying

$$\bar{y}''' + 2P\bar{y}'' + (Q + P' + P^2)\bar{y}' + (Q' - R + QP)\bar{y} = 0$$

where as shown in [3] we have $a = Q - b' - b(P - b)$

Example 2.3. *Consider the differential equation below:*

$$y^{''} + 5xy^{'} - 10y = 0, \tag{39}$$

Our goal is to derive an associated third order equation with two solutions equal to the two solutions of equation 39. The integrating factor associated to 39 will be given by $\mu = e^{\frac{5}{2}x^2}$ *which results in the two solutions* $y_1 = (x^2 + \frac{1}{5})$ *and* $y_2 = (x^2 + \frac{1}{5}) \int \frac{e^{\frac{-5}{2}x^2}}{(x^2+\frac{1}{5})^2} dx$. *Subsequently we wish to find* $\bar{P}$, $\bar{Q}$ *and* R *such that*

$$y^{'''} + \bar{P}y^{''} + \bar{Q}y^{'} + Ry = 0, \tag{40}$$

has two solutions equal to y_1 *and* y_2 *above. Recall the expression*

$$\frac{\mu'}{\mu} = \bar{P} - b = \frac{\bar{Q} - b' - a}{b} = \frac{R - a'}{a}$$

where $a = -10$ *and* $b = 5x$.
We have $\frac{\mu'}{\mu} = 5x$ *which implies that* $\bar{P} - 5x = 5x$ *and hence* $\bar{P} = 10x$. *Also,* $\frac{\bar{Q} - b' - a}{b} = 5x$ *imples* $\frac{\bar{Q} - 5 + 10}{5x} = 5x$ *and hence* $\bar{Q} = 25x^2 - 5$. *Finally,*

$\frac{R-a'}{a} = 5x$ *implies* $\frac{R-0}{-10} = 5x$ *and hence* $R = -50x$. *Thus, equation* 40 *becomes*

$$y''' + 10xy'' + (25x^2 - 5)y' + -50xy = 0, \tag{41}$$

which will be the associated third order differential equation. Recall that from [3] *equation* 41 *can be rewritten as*

$$(\mu(y'' + by' + ay))' = 0, \tag{42}$$

with $\mu = e^{\frac{5}{2}x^2}$, $a = -10$ *and* $b = 5x$.

Proof. Expanding $(\mu(y'' + by' + ay))'$ results in

$$\mu(y''' + by'' + b'y' + a'y + ay') + \mu'(y'' + by' + ay)$$

which simplifies to

$$y''' + by'' + (a + b')y' + a'y = -\frac{\mu'}{\mu}(y'' + by' + ay).$$

Recalling that $\frac{\mu'}{\mu} = 5x$ and substituting the values of a and b results in

$$y''' + 5xy'' + (-5)y' + 0 = -5x(y'' + 5xy' - 10y).$$

which then simplifies to

$$y''' + 10xy'' + (25x^2 - 5)y' + -50xy = 0$$

□

Now, integrating 42 *and simplifying results in the non-homogeneous equation below, we obtain:*

$$(y'' + 5xy' - 10y') = Ce^{-\frac{5}{2}x^2}.$$

We then see that the homogeneous part of this equation is equal to the homogeneous equation in 39 *which implies that the homogeneous solutions of equation* 41 *are the same as the solutions for the differential equation in equation* 39.

Therefore, using the methodology in [2] *we can conclude that the solution of the third order order differential equation:*

$$y''' + 10xy'' + (25x^2 - 5)y' + -50xy = 0$$

is given by:

$$c(x^2+\frac{1}{5})\int\frac{e^{-\frac{5}{2}x^2}}{(x^2+\frac{1}{5})^2}(\frac{x^3}{3}+\frac{1}{5}x)dx+k(x^2+\frac{1}{5})\int\frac{e^{-\frac{5}{2}x^2}}{(x^2+\frac{1}{5})^2}dx+l(x^2+\frac{1}{5})$$

We recall that:

$$k(x^2+\frac{1}{5})\int\frac{e^{-\frac{5}{2}x^2}}{(x^2+\frac{1}{5})^2}dx+l(x^2+\frac{1}{5})$$

is a solution of:

$$y^{''}+5xy^{'}-10y=0$$

Thus we have shown that given a third order differential equation

$$y^{'''}+\bar{P}y^{''}+\bar{Q}y^{'}+Ry=0$$

and a second order differential equation

$$y^{''}+by^{'}+ay=0$$

such that the following relation holds $\bar{P}=b+\frac{\mu'}{\mu}$, $\bar{Q}=b(\frac{\mu'}{\mu})+a+b^{'}$ *and* $R=a(\frac{\mu'}{\mu})+a'$ *where* μ *is the integrating factor for the second order equation, the two homogeneous solutions for the second order differential equation are also homogenuos solutions for the third order differential equation.*

2.3 Fourth Order Ordinary Differential Equations.

Consider the fourth order ODE below:

$$y^{iv}+3xy^{'''}+12y^{''}=0 \tag{43}$$

Now equation 43 can be rewritten as

$$(y^{'''}+3xy^{''}+9y^{'})^{'}=0,$$

which after integrating both sides yields

$$y^{'''}+3xy^{''}+9y^{'}=C.$$

Again the above equation can be rewritten as

$$(y^{''}+3xy^{'}+3y)^{'})^{'}=C,$$

which yields

$$(y^{''} + 3xy^{'} + 3y)^{'} = Cx + D$$

after integration. Further simplification now will result in

$$(y^{'} + 3xy)^{'} = Cx + D.$$

Which again after integration yields

$$y^{'} + 3xy = C\frac{x^2}{2} + Dx + F.$$

The above is now a linear equation which we can solve using the integrating factor technique.
For this linear equation the integrating factor is $\mu = e^{\frac{3x^2}{2}}$.
Now multiplying through with μ we get

$$y^{'}e^{\frac{3x^2}{2}} + 3xye^{\frac{3x^2}{2}} = e^{\frac{3x^2}{2}}\left(C\frac{x^2}{2} + Dx + F\right),$$

which after solving yields

$$y(x) = He^{\frac{-3x^2}{2}} + e^{\frac{-3x^2}{2}}\int\left(\frac{C}{2}x^2e^{\frac{3x^2}{2}} + Dxe^{\frac{3x^2}{2}} + Fe^{\frac{3x^2}{2}}\right) \tag{44}$$

Thus the solution for y above is a solution for equation 43.
On the other hand if we used the change of variable $v = y'$, equation 43 becomes:

$$v^{'''} + 3xv^{''} + 12v^{'} = 0 \tag{45}$$

and a change of variable using $u = v^{'}$, yields:

$$u^{''} + 3xu^{'} + 12u = 0 \tag{46}$$

And clearly we can deduce that $y(x) = \int v = \int\int u$

3 Generalized Solutions to Even and Odd Self-Adjoint DE's

In this section we present generalized methodology for even and odd order self-adjoint DE's respectively. From [3] we have the solutions for the second and third order self-adjoint DE. We present the solution to the 4th and 5th order and derive a generalization from the solutions.

3.1 Second-Order Self-Adjoint DEs

Theorem 1. *If a second-order self-adjoint ODE*

$$(r(x)y')' + q(x)y = 0 \tag{47}$$

verifies the condition $q(x) = \frac{r''}{2} - \frac{(r')^2}{4r}$, *then the solution to* (47) *is*

$$y(x) = \frac{1}{\sqrt{r(x)}}\left(C_1 x + C_2\right) \tag{48}$$

where $r(x) > 0,\ q(x)$ *are continuous differentiable functions and* $C_1,\ C_2$ *are arbitrary constants.*

3.2 Third-Order Self-Adjoint DEs

Theorem 2. *If a third-order self-adjoint ODE*

$$(r(x)y')'' + (q(x)y)' + p(x)y = 0 \tag{49}$$

verifies the conditions $q = r'' - \frac{2(r')^2}{3r}$ and $p = -\frac{r'''}{3} + \frac{2r'r''}{3r} - \frac{10(r')^3}{27r^2}$, *then the solution to* (49) *is*

$$y(x) = \frac{1}{\sqrt[3]{r^2}}\left(C_1 x^2 + C_2 x + C_3\right) \tag{50}$$

where $r(x) > 0,\ p(x),\ q(x)$ *are continuous differentiable functions and* $C_1,\ C_2,\ C_3$ *are arbitrary constants.*

Proofs for the theorems above can be found in [**?**]

3.3 Fourth-Order Self-Adjoint DEs

Theorem 3. *If a fourth-order self-adjoint ODE*

$$(r(x)y'')'' + (q(x)y')' + p(x)y = 0 \tag{51}$$

verifies the conditions $q = 2r'' - \frac{3}{2}\frac{(r')^2}{r}$ *and* $p = \frac{1}{16}\left[8r^{(4)} - \frac{16}{r}r'r''' - \frac{12}{r}(r'')^2 + \frac{36}{r^2}(r')^2 r'' - \frac{15}{r^3}(r')^4\right]$, *then the solution to* (51) *is*

$$y(x) = \frac{1}{\sqrt{r}}\left(C_1 x^3 + C_2 x^2 + C_3 x + C_4\right), \tag{52}$$

where $r(x) > 0,\ p(x),\ q(x)$ *are continuous differentiable functions, and* $C_1,\ C_2,\ C_3,\ C_4$ *are arbitrary constants.*

Proof. The fourth order self-adjoint ODE

$$(r(x)y'')'' + (q(x)y')' + p(x)y = 0$$

can be written as:

$$ry^{(4)} + 2r'y''' + (r'' + q)y'' + q'y' + py = 0,$$

which implies that

$$y^{(4)} + \frac{2r'}{r}y''' + \frac{(r'' + q)}{r}y'' + \frac{q'}{r}y' + \frac{p}{r}y = 0, \tag{53}$$

Considering a change of variable for $y = u(x)v(x)$ in (53), where $u(x)$ and $v(x)$ are continuous and differentiable functions, we obtain:

$$y' = (uv)' = u'v + uv'$$
$$y'' = (uv)'' = u''v + 2u'v' + uv''$$
$$y''' = (uv)''' = u'''v + 3u''v' + 3u'v'' + uv'''$$
$$y^{(4)} = (uv)^4 = u^{(4)}v + 4u'''v' + 6u''v'' + 4u'v''' + uv^{(4)}$$

$$u^{(4)}v + u'''\left(4v' + \frac{2r'}{r}v\right) + u''\left(6v'' + \frac{6r'}{r}v' + \frac{(r'' + q)}{r}v\right) +$$
$$u'\left(4v''' + \frac{6r'}{r}v'' + \frac{2(r'' + q)}{r}v' + \frac{q'}{r}v\right) + u\left(v^{(4)} + \frac{2r'}{r}v''' + \frac{(r'' + q)}{r}v'' + \frac{q'}{r}v' + \frac{p}{r}v\right) = 0$$

Assuming that the coefficient of u''' is zero, we can solve for $v(x)$ as follows:

$$4v' + \frac{2r'}{r}v = 0. \tag{54}$$

Then, without loss of generality, the corresponding solution to (54) is

$$v(x) = e^{-\frac{1}{2}\int \frac{r'}{r}dx} = r^{-\frac{1}{2}} = \frac{1}{\sqrt{r}} \tag{55}$$

$$v'(x) = -\frac{1}{2}r^{-\frac{3}{2}}r' \tag{56}$$

$$v''(x) = \frac{3}{4}r^{-\frac{5}{2}}(r')^2 - \frac{1}{2}r^{-\frac{3}{2}}r'' \tag{57}$$

$$v'''(x) = -\frac{15}{8} r^{-\frac{7}{2}} (r')^3 + \frac{9}{4} r^{-\frac{5}{2}} r' r'' - \frac{1}{2} r^{-\frac{3}{2}} r''' \tag{58}$$

$$v^{(4)}(x) = \frac{105}{16} r^{-\frac{9}{2}} (r')^4 - \frac{45}{4} r^{-\frac{7}{2}} (r')^2 r'' + \frac{9}{4} r^{-\frac{5}{2}} (r'')^2 + 3 r^{-\frac{5}{2}} r' r''' - \frac{1}{2} r^{-\frac{3}{2}} r^{(4)} \tag{59}$$

Next, assuming that the coefficients of u'', u' and u are all zero we have:

$$6v'' + \frac{6r'}{r} v' + \frac{(r'' + q)}{r} v = 0, \tag{60}$$

$$4v''' + \frac{6r'}{r} v'' + \frac{2(r'' + q)}{r} v' + \frac{q'}{r} v = 0 \tag{61}$$

and

$$v^{(4)} + \frac{2r'}{r} v''' + \frac{(r'' + q)}{r} v'' + \frac{q'}{r} v' + \frac{p}{r} v = 0 \tag{62}$$

By substituting (55) into (60) and (62) we obtain:

$$6\left(\frac{3}{4} r^{-\frac{5}{2}} (r')^2 - \frac{1}{2} r^{-\frac{3}{2}} r''\right) + \frac{6r'}{r}\left(-\frac{1}{2} r^{-\frac{3}{2}} r'\right) + \frac{r'' + q}{r}\left(r^{-\frac{1}{2}}\right) = 0$$

$$\begin{gathered}
\left(\frac{105}{16} r^{-\frac{9}{2}} (r')^4 - \frac{45}{4} r^{-\frac{7}{2}} (r')^2 r'' + \frac{9}{4} r^{-\frac{5}{2}} (r'')^2 + 3 r^{-\frac{5}{2}} r' r''' - \frac{1}{2} r^{-\frac{3}{2}} r^{(4)}\right) \\
+ \frac{2r'}{r}\left(-\frac{15}{8} r^{-\frac{7}{2}} (r')^3 + \frac{9}{4} r^{-\frac{5}{2}} r' r'' - \frac{1}{2} r^{-\frac{3}{2}} r'''\right) \\
+ \left(\frac{r'' + q}{r}\right)\left(\frac{3}{4} r^{-\frac{5}{2}} (r')^2 - \frac{1}{2} r^{-\frac{3}{2}} r''\right) + \frac{q'}{r}\left(-\frac{1}{2} r^{-\frac{3}{2}} r'\right) \\
+ \frac{p}{r}\left(r^{-\frac{1}{2}}\right) = 0
\end{gathered} \tag{63}$$

Simplifying these substitutions we obtain:

$$\frac{3}{2r} (r')^2 - 2r'' + q = 0 \tag{64}$$

and

$$-\frac{1}{2} r^{(4)} + \frac{r' r'''}{r} + \frac{3}{4r} (r'')^2 - \frac{9}{4r^2} (r')^2 r'' + \frac{15}{16 r^3} (r')^4 + p = 0 \tag{65}$$

We obtain corresponding conditions for q and p from (64) and (65) as follows:

$$q = 2r'' - \frac{3}{2}\frac{(r')^2}{r} \tag{66}$$

$$p = \frac{1}{16}\left(8r^{(4)} - \frac{16}{r}r'r''' - \frac{12}{r}(r'')^2 + \frac{36}{r^2}(r')^2r'' - \frac{15}{r^3}(r')^4\right) \tag{67}$$

From the equations (54), (60), (62), (66) and (67), we obtain:

$$vu^{(4)} = 0 \implies u^{(4)} = 0 \tag{68}$$

and

$$u(x) = C_1x^3 + C_2x^2 + C_3x + C_4 \tag{69}$$

where $C_1,\ C_2,\ C_3$ and C_4 are arbitrary constants. Hence, the solution of the fourth-order self-adjoint differential equation (51) is as follows:

$$y(x) = u(x)v(x) \tag{70}$$

and we conclude that

$$y(x) = \frac{1}{\sqrt{r}}\left(C_1x^3 + C_2x^2 + C_3x + C_4\right) \tag{71}$$

3.4 Fourth-Order Material Science Application

The paper [18] provides an application of fourth-order self-adjoint ordinary differential equations to deformations of isotropic incompressible hyperelastic materials. In general, Hill considers

$$[f_0(R)u'']'' + [f_1(R)u']' + f_2(R)u = 0,$$

where the solution u is a function of R, the functions $f_0(R),\ f_1(R),\ f_2(R)$ are sufficiently continuously differentiable and also allow the self-adjoint property of the differential operator. In particular, Hill defines the second order self-adjoint differential operator D^2 by

$$D^2u = [a(R)u']' + b(R)u,$$

where a and b are functions of R. Considering another function c, then the fourth order equation can be written

$$D^2[c(R)D^2u] = 0,$$

and it is self-adjoint if the function f_0, f_1, and f_2 satisfy

$$f_0(R) = a^2 c \tag{72}$$
$$f_1(R) = a(ca')' + 2abc, \tag{73}$$
$$f_2(R) = b^2 c + [a(bc)']'. \tag{74}$$

While the stability of solutions is investigated in [18], analytic solutions are generally not proposed, remarking that there are a few rare scenarios where analytic solutions can be determined by "inspection." In the paper, the stability of solutions relating to the simultaneous inflation and extension of a cylindrical tube, the symmetrical expansion of a spherical shell, and plane straightening and stretching of a sector of a circular-cylindrical tube is considered through ad-hoc factorization. Having a method to analytically solve such high order differential equations will definitely give insight into more precise properties of the solutions, and thereby open material sciences to new opportunities.

3.5 Fifth-Order Self-Adjoint DEs

Theorem 4. *If a fifth-order self-adjoint ODE*

$$(r(x)y''')'' + (q(x)y'')' + (p(x)y)' = 0 \tag{75}$$

verifies the conditions $q = 3r'' - \frac{12}{5}\frac{(r')^2}{r}$ *and* $p = -\frac{2}{125r^3}\left[384(r')^4 - 840r(r')^2r'' + 225r^2(r'')^2 + 350r^2r'r''' - 125r^3r^{(4)}\right]$, *then the solution* to (75) *is*

$$y(x) = \frac{1}{\sqrt[5]{r^2}}\left(C_1x^4 + C_2x^3 + C_3x^2 + C_4x + C_5\right), \tag{76}$$

where $r(x) > 0,\ p(x),\ q(x)$ *are continuous differentiable functions, and* $C_1,\ C_2,\ C_3,\ C_4,\ C_5$ *are arbitrary constants.*

Proof. The fifth order self-adjoint ODE

$$(r(x)y''')'' + (q(x)y')' + p(x)y = 0$$

can be written as:

$$ry^{(5)} + 2r'y^{(4)} + (r'' + q)y''' + q'y'' + py' + p'y = 0$$

which implies that

$$y^{(5)} + \frac{2r'}{r}y^{(4)} + \frac{(r''+q)}{r}y''' + \frac{q'}{r}y'' + \frac{p}{r}y' + \frac{p'}{r}y = 0, \tag{77}$$

Considering a change of variable for $y = u(x)v(x)$ in (77), where $u(x)$ and $v(x)$ are continuous and differentiable functions, we obtain

$$y' = (uv)' = u'v + uv'$$

$$y'' = (uv)'' = u''v + 2u'v' + uv''$$

$$y''' = (uv)''' = u'''v + 3u''v' + 3u'v'' + uv'''$$

$$y^{(4)} = (uv)^4 = u^{(4)}v + 4u'''v' + 6u''v'' + 4u'v''' + uv^{(4)}$$

$$y^{(5)} = (uv)^5 = u^{(5)}v + 5u^{(4)}v' + 10u'''v'' + 10u''v''' + 5u'v^{(4)} + uv^{(5)}$$

$$u^{(5)}v + \left(5v' + \frac{2r'}{r}v\right)u^{(4)} + \left(10v'' + \frac{8r'}{r}v' + \frac{(r''+q)}{r}v\right)u'''$$

$$+\left(10v''' + \frac{12r'}{r}v'' + 3\frac{(r''+q)}{r}v' + \frac{q'}{r}v\right)u'' + \left(5v^{(4)} + \frac{8r'}{r}v''' + 3\frac{(r''+q)}{r}v'' + \frac{2q'}{r}v' + \frac{p}{r}v\right)u'$$

$$+\left(v^{(5)} + \frac{2r'}{r}v^{(4)} + \frac{(r''+q)}{r}v''' + \frac{q'}{r}v'' + \frac{p}{r}v' + \frac{p'}{r}v\right)u = 0$$

Assuming that the coefficient of $u^{(4)}$ is zero, we can solve for $v(x)$ as follows:

$$5v' + \frac{2r'}{r}v = 0. \tag{78}$$

Then, without loss of generality, the corresponding solution to (78) is

$$v(x) = e^{-\frac{2}{5}\int \frac{r'}{r}dx} = r^{-\frac{2}{5}} = \frac{1}{\sqrt[5]{r^2}} \tag{79}$$

$$v'(x) = -\frac{2r'}{5r^{-\frac{7}{5}}} \tag{80}$$

$$v''(x) = \frac{14(r')^2}{25r^{\frac{12}{5}}} - \frac{2r''}{5r^{\frac{7}{5}}} \tag{81}$$

$$v'''(x) = -\frac{168(r')^3}{125r^{\frac{17}{5}}} + \frac{42r'r''}{25r^{\frac{12}{5}}} - \frac{2r'''}{5r^{\frac{7}{5}}} \tag{82}$$

$$v^{(4)}(x) = \frac{2856(r')^4}{625r^{\frac{22}{5}}} - \frac{1008(r')^2r''}{125r^{\frac{17}{5}}} + \frac{42(r'')^2}{25r^{\frac{12}{5}}} + \frac{56r'r'''}{25r^{\frac{12}{5}}} - \frac{2r^{(4)}}{5r^{\frac{7}{5}}} \tag{83}$$

$$v^{(5)}(x) = -\frac{62832(r')^5}{3125r^{\frac{27}{5}}} + \frac{5712(r')^3r''}{125r^{\frac{22}{5}}} - \frac{504r'(r'')^2}{25r^{\frac{17}{5}}} - \frac{336(r')^2r'''}{25r^{\frac{17}{5}}} + \frac{28r''r'''}{5r^{\frac{12}{5}}} + \frac{14r'r^{(4)}}{5r^{\frac{12}{5}}} - \frac{2r^{(5)}}{5r^{\frac{7}{5}}} \tag{84}$$

Assuming that the coefficients of $u''', u'',\ u'$ and u are all zero, we obtain

$$10v'' + \frac{8r'}{r}v' + \frac{(r''+q)}{r}v = 0 \tag{85}$$

$$10v''' + \frac{12r'}{r}v'' + 3\frac{(r''+q)}{r}v' + \frac{q'}{r}v = 0 \tag{86}$$

$$5v^{(4)} + \frac{8r'}{r}v''' + 3\frac{(r''+q)}{r}v'' + \frac{2q'}{r}v' + \frac{p}{r}v = 0 \tag{87}$$

$$v^{(5)} + \frac{2r'}{r}v^{(4)} + \frac{(r''+q)}{r}v''' + \frac{q'}{r}v'' + \frac{p}{r}v' + \frac{p'}{r}v = 0 \tag{88}$$

Substituting $v, v', v'', v''', v^{(4)}$ and $v^{(5)}$ into equations (85) - (88), we get:

$$10\left[\frac{14(r')^2}{25r^{\frac{12}{5}}} - \frac{2r''}{5r^{\frac{7}{5}}}\right] + \frac{8r'}{r}\left[-\frac{2r'}{5r^{-\frac{7}{5}}}\right] + \frac{(r''+q)}{r}\left[r^{-\frac{2}{5}}\right] = 0 \tag{89}$$

Simplifying equation (89) gives

$$q = 3r'' - \frac{12(r')^2}{5r}$$

Also, from equation (87) we get:

$$5\left[\frac{2856(r')^4}{625r^{\frac{22}{5}}} - \frac{1008(r')^2r''}{125r^{\frac{17}{5}}} + \frac{42(r'')^2}{25r^{\frac{12}{5}}} + \frac{56r'r'''}{25r^{\frac{12}{5}}} - \frac{2r^{(4)}}{5r^{\frac{7}{5}}}\right] + \frac{8r'}{r}\left[-\frac{168(r')^3}{125r^{\frac{17}{5}}} + \frac{42r'r''}{25r^{\frac{12}{5}}} - \frac{2r'''}{5r^{\frac{7}{5}}}\right]$$
$$+ 3\frac{(r''+q)}{r}\left[\frac{14(r')^2}{25r^{\frac{12}{5}}} - \frac{2r''}{5r^{\frac{7}{5}}}\right] + \frac{2q'}{r}\left[-\frac{2r'}{5r^{-\frac{7}{5}}}\right] + \frac{p}{r}\left[r^{-\frac{2}{5}}\right]. \tag{90}$$

Simplifying equation (90) we obtain:

$$p = -\frac{2}{125r^3}\left[384(r')^4 - 840r(r')^2r'' + 225r^2(r'')^2 + 350r^2r'r''' - 125r^3r^{(4)}\right]$$

From

$$u^{(5)}v = 0 \implies u^{(5)} = 0 \tag{91}$$

and

$$u(x) = C_1x^4 + C_2x^3 + C_3x^2 + C_4x + C_5 \tag{92}$$

where $C_1,\ C_2,\ C_3,\ C_4$ and C_5 are arbitrary constants. Hence, the solution of the fifth-order self-adjoint differential equation (75) is as follows:

$$y(x) = u(x)v(x) \tag{93}$$

and we conclude that:

$$y(x) = \frac{1}{\sqrt[5]{r^2}}\left(C_1x^4 + C_2x^3 + C_3x^2 + C_4x + C_5\right) \tag{94}$$

3.6 nth-Order Self-Adjoint DEs

From above theorems and proofs, we are able to generalize solutions of self-adjoint ODEs based on the order for continuous differentiable functions $r(x) > 0,\ p(x),\ q(x)$.

3.6.1 Even n

An even nth-order self-adjoint ODE

$$[r(x)y^{\{\frac{n}{2}\}}]^{\{\frac{n}{2}\}} + [q(x)y^{\{\frac{n}{2}-1\}}]^{\{\frac{n}{2}-1\}} + [p(x)y^{\{\frac{(n-1)}{2}-2\}}]^{\{\frac{(n-1)}{2}-2\}} + ... + t(x)y = 0, \tag{95}$$

has general solution

$$y(x) = \frac{1}{\sqrt{r(x)}}\left(C_1x^{n-1} + C_2x^{n-2} + C_3x^{n-3} + ... + C_n\right), \tag{96}$$

where $C_1,\ C_2,\ C_3,\ ...,\ C_n$ are arbitrary constants, if corresponding conditions of continuous differentiable functions $r(x) > 0,\ p(x),\ q(x),\ e.t.c$ are verified.

3.6.2 Odd n

An odd nth-order self-adjoint ODE

$$[r(x)y^{\{\frac{(n-1)}{2}+1\}}]^{\{\frac{(n-1)}{2}\}} + [q(x)y^{\{\frac{(n-1)}{2}\}}]^{\{\frac{(n-1)}{2}-1\}} + [p(x)y^{\{\frac{(n-1)}{2}-1\}}]^{\{\frac{(n-1)}{2}-2\}} + ... + t(x)y = 0, \tag{97}$$

has general solution

$$y(x) = \frac{1}{\sqrt[n]{r(x)}}\left(C_1x^{n-1} + C_2x^{n-2} + C_3x^{n-3} + ... + C_n\right), \tag{98}$$

where $C_1,\ C_2,\ C_3,\ ...,\ C_n$ are arbitrary constants, if corresponding conditions of continuous differentiable functions $r(x) > 0,\ p(x),\ q(x),$ e.t.c are verified.

4 Conclusion

In this work we have used the integrating factor approach for solving higher order differential equations. We show that the solutions to a second order differential equation are also solutions to an associated third order differential equation of the Ricatti type, and we presented examples. In addition, we developed a self-adjoint type formulation for solving higher order ODEs including odd and even order ODEs. The generalized methodologies presented may serve as a reference for solving higher order ODEs. Transformation of differential equation to self-adjoint differential equation is based on the choice of $q(x)$ and $p(x)$.

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