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SOLVING THIRD ORDER ORDINARY DIFFERENTIAL EQUATIONS BY USING RICATTI EQUATIONS

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Synopsis:

This paper deals with an analytic approach for solving third order ordinary differential equations (ODEs). We show that a general linear differential equation has an associated Ricatti type equation. In addition, the coefficients of the differential equation associated to the integrating factor for the original equation verify Ricatti type equations. The solutions for these Ricatti differential equations are related to the solutions of the original differential equation.

Solving Third Order Ordinary Differential Equations by Using Ricatti Equations

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Abstract

This paper deals with an analytic approach for solving third order ordinary differential equations (ODEs). We show that a general linear differential equation has an associated Ricatti type equation. In addition the coefficients of the differential equation associated to the integrating factor for the original equation verify Ricatti type equations. The solutions for these Ricatti differential equations are related to the solutions of the original differential equation.

Keywords: Third Order ODE; Integrating Factor technique; Ricatti ODE;

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1 Introduction

Several modeling techniques have been developed for solving differential equations (DEs) by considering its different characteristics. These characteristics indicate the type of DEs like ordinary differential equations (ODEs), partial differential equations (PDEs) and stochastic differential equations (SDEs). The various types of differential equations have been studied with application to several disciplines such as physics [1], finance [8], geophysics [9] and biology [2], just to mention a few.

In general, solving higher order DEs is complex and numerical methods are usually needed to solve these equations with initial or boundary conditions. For example, Khawaja et al. (2018) [5] used iterative power series of $sech(x)$ to solve nonlinear ordinary differential equations (ODEs) . Vitoriano (2016) [6] also used the finite element method to solve PDEs. Mariani and Tweneboah (2016); Mariani et al. (2016, 2018) used Itô's calculus to solve stochastic differential equations (SDEs) (see [8, 9, 12]). The above studies suggest that different numerical techniques are needed when solving complex DEs.

Leighton and Nehari (1958) [13] studied the oscillation properties of the solutions of fourth order self-adjoint DEs i.e.,

$$(r(x)y'')'' + (q(x)y')' + p(x)y = 0, \tag{1.1}$$

where $r(x) > 0$ and $p(x) > 0$. In particular, the authors studied the case where the functions $r(x)$ and $p(x)$ do not change sign in $(0, \infty)$. Barret [15] investigated a special case of Label (1.1), where $q(x) = 0$ and both $r(x)$ and $p(x)$ are positive and continuous on $(0, \infty)$. The author extended this with the object of paralleling the self-adjoint DE of the second order with positive and continuous coefficients. In the paper by [11], the authors used change of variables to find the analytic solution to some self-adjoint DEs of the Fourth Order. In general, the solutions to self-adjoint equations are complex, hence we most often tend to solve them using numerical methods, which contain approximation errors. The solution technique proposed in this paper produces analytic solutions unlike numerical methods that have approximation errors. In Ref. [7], the authors extended the concept of self-adjoint equations to solve higher order differential equations including odd orders, arguing that this work may serve as a reference for solving other higher order self-adjoint type ODEs. According to the authors' knowledge, no comprehensive work was dedicated to solving odd order self-adjoint type ODEs. Indeed, at present, general research in self-adjoint type ODEs are concerned with solving even orders ODEs (see [4, 13, 11], etc.). In addition, little work has been carried out to study integrating factor type techniques for solving higher order ODEs and previous works have only studied the integrating factor technique for solving first order ODEs [14]. An integrating factor is a function by which an ordinary differential equation can be multiplied in order to make it integrable.

In this paper, we provide an analytic method for solving second and third order differential equations (ODEs), without the need of solving them numerically. The paper is outlined as follows: In Section 2, we briefly present some concepts and definitions that will be used throughout this paper. In Section 3, we present the integrating factor theory for solving third order ODEs. Conclusions are contained in Section 5.

2 Second Order ODEs and Ricatti equations

In this work, we will present an integrating factor methodology for solving ODEs. The aim is to develop a technique for solving higher order differential equations. First, we present the definition of a Ricatti ODE of the first order.

Definition 2.1. *A Ricatti ODE of the first order is any equation of the form*

$$y' + P(x) + Q(x)y = R(x)y^n, \tag{2.1}$$

where $P(x), Q(x)$ *and* $R(x)$ *are continuous functions and* $n > 1$ *is a real number.*

In order to solve a second order ODE, we propose a first order Ricatti ODE for a variable b. In fact, this Ricatti equation is obtained by using the coefficients of the second order ODE and the particular solution for this equation helps to obtain the integrating factor of the second order ODE. Using integrating type techniques, we obtain a particular solution of the second order differential equation. We proceed with the methodology as follows.

Consider a general second order ODE,

$$y'' + P(x)y' + Q(x)y = R(x), \tag{2.2}$$

where $P(x), Q(x)$ and $R(x)$ are continuous functions.

We can express the second order ODE as:

$$(u(y' + by))' = uR(x), \tag{2.3}$$

where u is an integrating factor. This equation can be rewritten as:

$$y'(u' + ub) + y(u'b + ub') + uy'' = uR(x). \tag{2.4}$$

From Labels (2.2) and (2.4), we can compare the coefficients as follows:

$$u' + ub = uP(x), \tag{2.5}$$

$$u'b + ub' = uQ(x). \tag{2.6}$$

We then simplify the above two equations and obtain the Ricatti equation for b:

$$b' = Q(x) - bP(x) + b^2. \tag{2.7}$$

If we know the homogeneous solution y_1 of (2.2), then $-\frac{y_1'}{y_1} = b$ is a solution of the Ricatti equation $b' = b^2 + Q - bP$ and if u in (2.5) is given by $\frac{u'}{u} = P - b$, we obtain that u is an integrating factor of (2.2) and this allows us to obtain a particular solution. We also observe that, from (2.5) and (2.6), it is possible to obtain the second order differential equation for the integrating factor, which is $u'' - u'P + u(Q - P') = 0$. As $ub = uP - u'$, it implies that $(ub)' = uQ = u'P + uP' - u''$. Therefore, $u'' - u'P + u(Q - P') = 0$.

Next, we present an example.

Example 2.2. *Consider a second order ODE,*

$$y'' - xy' + y = x^k, \tag{2.8}$$

where k is any even number.

In particular, we consider the case where $k = 2$ i.e.,

$$y'' - xy' + y = x^2. \tag{2.9}$$

Comparing this equation with (2.2)*, we observe that $P = -x$, $Q = 1$, and $R = x^2$.*

The Ricatti equation for b is given by:

$$b' = Q - bP + b^2. \tag{2.10}$$

Knowing that $y_1 = x$ is a solution of the homogeneous equation associated with (2.9) *i.e., $y'' - xy' + y = 0$, as $b = -\frac{y_1'}{y_1} = -\frac{1}{x}$, then $b = -\frac{1}{x}$ is a particular solution of (2.10). We now compare between Equations (2.2) and (2.4) and obtain:*

$$\frac{u'}{u} = P - b.$$

Then,

$$\frac{u'}{u} = -x + \frac{1}{x}. \tag{2.11}$$

Integrating both sides of (2.11)*, we have*

$$\ln u = -\frac{x^2}{2} + lnx + C.$$

Without loss of generality, $u = e^{-x^2/2}x$.

Now plugging u, b and R into (2.3) *and integrating both sides, we obtain*

$$xe^{-x^2/2}\left(y' - \frac{1}{x}y\right) = \int x^3 e^{-x^2/2} dx = -e^{-\frac{x^2}{2}}(x^2 + 2) + C.$$

This implies that

$$y' - \frac{1}{x}y = -\left(x + \frac{2}{x}\right) + \frac{Ce^{x^2/2}}{x}.$$

The solution to this last equation can be found by using first degree order integrating factor theory.

3 Integrating Factor Technique for Third Order ODEs

We begin this subsection with a theorem [7].

Theorem 3.1. *Given*

$$y''' + P(x)y'' + Q(x)y' + R(x)y = f(x), \tag{3.1}$$

if we know a solution to the associated integrating factor equation

$$u''' - Pu'' + (Q - 2P')u' + (Q' - P'' - R)u = 0 \tag{3.2}$$

or, alternatively, a solution to

$$y''' + 2Py'' + (P' + P^2 + Q)y' + (Q' - R + QP)y = 0, \tag{3.3}$$

then we can find a particular solution to (3.1)*.*

Proof. We solve (3.1) by formulating

$$[u(y'' + by' + ay)]' = uf(x), \tag{3.4}$$

where u satisfies (3.2). Multiplying through (3.1) by u, we obtain

$$u(y''' + P(x)y'' + Q(x)y' + R(x)y) = uf(x). \tag{3.5}$$

From (3.4) and (3.5), we get

$$u'(y'' + by' + ay) + u(y'' + by' + ay)' = uf(x). \tag{3.6}$$

Expanding, we get

$$y'''u + y''(u' + ub) + y'(u'b + ub' + ua) + y(u'a + ua') = uf(x). \tag{3.7}$$

From (3.4)-(3.7), we obtain the following three conditions:

(i) $uP = u' + ub$,

(ii) $uQ = u'b + ub' + ua$,

(iii) $uR = u'a + ua'$.

From (i), we have that:

$$\frac{u'}{u} = P - b \tag{3.8}$$

and from (ii):

$$\frac{u'}{u} = \frac{Q - b' - a}{b}. \tag{3.9}$$

Thus, from (3.8) and (3.9), $P - b = \frac{Q-b'-a}{b}$; then, we have that $(P - b)b = Q - b' - a$ and, solving for a, we obtain

$$a = Q - b' - b(P - b). \tag{3.10}$$

Finally from (iii), we have $u'a = u(R - a')$ and so

$$\frac{u'}{u} = \frac{R - a'}{a}. \tag{3.11}$$

Combining with (3.8), we obtain

$$\frac{u'}{u} = \frac{R - a'}{a} = P - b. \tag{3.12}$$

Thus, $a(P - b) = R - a'$. Replacing in (3.12) a and a', we get

$$[Q - b' - b(P - b)]\,(P - b) = R - [Q - b' - b(P - b)]'. \tag{3.13}$$

From (3.13), we obtain

$$b'' = Q' - R + QP + b(-Q - P' - P^2) + 3bb' - 2b'P + 2Pb^2 - b^3 \tag{3.14}$$

Using the change of variable $b = -\frac{y'}{y}$, we have that

$$b' = -\frac{y''}{y} + \frac{y'^2}{y^2}. \tag{3.15}$$

Then, replacing $b = -\frac{y'}{y}$ into (3.15), we conclude that $b' = -\frac{y''}{y} + b^2$, and

$$b'' = -\frac{y'''}{y} + \frac{y'}{y}\frac{y''}{y} + 2bb'. \tag{3.16}$$

Since $b' = -\frac{y''}{y} + b^2$, replacing $\frac{y'}{y} = -b$ into (3.16), we have

$$b'' = -\frac{y'''}{y} - b\frac{y''}{y} + 2bb'$$

and, using $b' = -\frac{y''}{y} + b^2$, we get $\frac{y''}{y} = b^2 - b'$; therefore,

$$b'' = -\frac{y'''}{y} - b^3 + 3bb'. \tag{3.17}$$

Considering the homogeneous differential equation associated with (3.1) i.e.,

$$y''' + P(x)y'' + Q(x)y' + R(x)y = 0,$$

we obtain

$$\frac{y'''}{y} = -P(x)\frac{y''}{y} - Q(x)\frac{y'}{y} - R(x). \tag{3.18}$$

Replacing $\frac{y''}{y} = b^2 - b'$ and $\frac{y'}{y} = -b$ in (3.18):

$$\frac{y'''}{y} = -P(x)\left[b^2 - b'\right] + Q(x)b - R(x)$$

and, replacing the last expression into (3.17), we get

$$b'' = R(x) + b^2P(x) + 3bb' - b^3 - bQ(x) - b'P(x). \tag{3.19}$$

Therefore, (3.19) represents the Ricatti equation associated with (3.1). On the other hand, if we replace (3.17) into (3.14) and use the relations $b = -\frac{y'}{y}$ and $b' = -\frac{y''}{y} + b^2$, we get:

$$y''' + 2Py'' + (P' + P^2 + Q)y' + (Q' - R + QP)y = 0, \tag{3.20}$$

which is the corresponding homogeneous equation associated with (3.14).

We recall from conditions (i) and (ii) that $ub = uP - u'$ and $(ub)' + ua = uQ$, respectively. Thus, from (i), $(ub)' = u'P + uP' - u''$ and replacing this into (ii), we have:

$$ua = uQ - u'P - uP' + u''. \tag{3.21}$$

From condition (iii) $(ua)' = uR$, so, taking the derivative of (3.21), we obtain the integrating factor equation

$$u''' - u''P + u'(Q - 2P') + u(Q' - P'' - R) = 0. \tag{3.22}$$

Thus, if we know one solution of (3.3), by using the change of variable $b = -\frac{y'}{y}$, we have one solution of (3.14) which means we have a solution for b, consequently for a from (i), (ii) and (iii). Hence the proof. □

3.1 Main results: Associated Ricatti type Equations

The following lemmas contain the main results.

Lemma 3.2. *Solution to the Ricatti type Equation for b:*
Considering the Ricatti type equation from (3.14)*, using as a corollary of the previous theorem we can conclude that by using $b = -\frac{y'}{y}$, if we have a*

solution of (3.20)*, then we have a solution of* (3.14)*. We also obtain* a *by using* $a = Q - b' - b(P - b)$*. If we have* a *and* b*, we are able to find the particular solution for the original equation for* (3.1)*. By using* (3.22)*, and* (3.12)*, we solve for* u *and hence obtain the particular solution of* (3.1) *and we can solve* (3.14)*.*

From equation (3.19)*, we can also conclude that if we know one solution of the homogeneous part of* (3.1)*, then using* $b = -\frac{y'}{y}$*, we can have the solution of* (3.19)*. Also from* (3.22)*, which is the equation of integrating factor of* (3.1) *and* (3.4)*, if we know one solution of* (3.22)*, using* (3.8)*, we find* b *and hence a solution of* (3.14)*.*

Lemma 3.3. *Solution to the Ricatti type Equation for* a*:*

From the above theorem we obtain

$$\frac{u'}{u} = \frac{R - a'}{a} = P - b. \tag{3.23}$$

Thus, $a(P - b) = R - a'$*. We get*

$$a' = R - a(P - b) \tag{3.24}$$

Combining (3.9) *and* (3.25)*,*

$$\frac{Q - b' - a}{b} = \frac{R - a'}{a}. \tag{3.25}$$

Thus, we obtain

$$b' = Q - a - \frac{R - a'}{a} b \tag{3.26}$$

Then from (3.23)*, replacing* $b = -\frac{R-a'}{a} + P$ *into* (3.26)*, we obtain a Ricatti type equation for* a *as follows:*

$$\left(\frac{a' - R}{a} + P\right)' = Q - a - \frac{(R - a')}{a}\left(\frac{a' - R}{a} + P\right). \tag{3.27}$$

Expanding equation (3.27) *above we get :*

$$-\left(\frac{R-a'}{a} - \frac{a''}{a}\right) + \frac{a'R}{a^2} - \frac{a'^2}{a^2} = Q - a + \frac{R^2}{a^2} - \frac{2Ra'}{a^2} + \left(\frac{a'}{a}\right)^2 - \frac{RP}{a} + \frac{a'P}{P} - P' \tag{3.28}$$

and simplifying equation (3.28) *we then obtain,*

$$a''a = a'(aP - 3R) + 2(a')^2 + a(R' - PR) + a^2(Q - P') + R^2 - a^3$$

which is the Ricatti type equation for a*.*

4 Examples

In this section, we present applications and examples of the associated Ricatti type equations.

Example 4.1. *Consider a third order ODE,*

$$y''' + 10xy'' + 25x^2y' - 25xy = 0. \tag{4.1}$$

The solutions for this homogenous third order ODE are

$$\begin{cases} y_1 = x \\ y_2 = e^{-\frac{5}{2}x^2} \\ y_3 = x \int \dfrac{e^{-\frac{5}{2}x^2}}{x^2} dx \end{cases} \tag{4.2}$$

Comparing (4.1) *with equation* (3.1) *we have* $P(x) = 10x$, $Q(x) = 25x^2$, $R(x) = -25x$ *and* $f(x) = 0$. *Then using the relation* $b = \frac{-y'}{y}$ *for example, if we take* $y_1 = x$ *then we have* $b = \frac{-y_1'}{y_1} = \frac{-1}{x}$ *satisfies the Ricatti equation* (3.19)

$$b'' = R(x) + b^2P(x) + 3bb' - b^3 - bQ(x) - b'P(x) \tag{4.3}$$

which in this case is:

$$b'' = -25x + \left(\frac{1}{x}\right)^2 10x + 3\left(\frac{-1}{x}\right)\left(\frac{1}{x^2}\right) - \left(-\frac{1}{x}\right)^3 - \left(\frac{-1}{x}\right) 25x^2 - \left(\frac{1}{x^2}\right) 10x \tag{4.4}$$

which is simplified as follows:

$$b'' = -25x + \frac{10}{x} - \frac{3}{x^3} + \frac{1}{x^3} + 25x - \frac{10}{x} = -\frac{2}{x^3} \tag{4.5}$$

Thus

$$b'' = -\frac{2}{x^3} \tag{4.6}$$

So we see that $b = -\frac{y'}{y}$ *satisfies* (4.3) *because* $b = -\frac{1}{x}$ *implies that* $b' = \frac{1}{x^2}$ *which also implies that* $b'' = -\frac{2}{x^3}$.

Example 4.2. *Consider the third order ODE discussed in the previous example.*

$$y''' + 10xy'' + 25x^2y' - 25xy = 0. \tag{4.7}$$

We know that using that equation corresponding to the formula

$$b'' = R(x) + b^2P(x) + 3bb' - b^3 - bQ(x) - b'P(x) \tag{4.8}$$

where $P(x) = 10x$, $Q(x) = 25x^2$ *and* $R(x) = -25x$. *So if the solution to* (4.7) *are*

$$\begin{cases} y_1 = x \\ y_2 = e^{-\frac{5}{2}x^2} \\ y_3 = x\int \dfrac{e^{-\frac{5}{2}x^2}}{x^2}dx \end{cases} \tag{4.9}$$

then the solution to (4.8) *are*

$$b_1 = -\frac{y_1'}{y_1} = \frac{(e^{-\frac{5}{2}x^2})'}{e^{-\frac{5}{2}x^2}} = 5x$$

$$b_2 = -\frac{y_2'}{y_2} = -\frac{1}{x}$$

and

$$b_3 = -\frac{y_3'}{y_3} = \frac{\left(x\int \dfrac{e^{-\frac{5}{2}x^2}}{x^2}dx\right)'}{x\int \dfrac{e^{-\frac{5}{2}x^2}}{x^2}dx} = -\frac{1}{x}\left[1 + \frac{1}{x}\frac{e^{-\frac{5}{2}x^2}}{\int \frac{e^{-\frac{5}{2}x^2}}{x^2}}\right]$$

So using the fact that b_1, b_2 *and* b_3 *are solutions to* (4.8), *and that on the other hand* (4.7) *with*

$$10x = 2P \tag{4.10}$$

$$25x^2 = Q + P' + P^2 \tag{4.11}$$

and

$$-25x = Q' - R + QP \tag{4.12}$$

which correspond to (3.20). *Solving for* (3.20) *using the relations from equations* (4.10)-(4.12) *we have that:*

$P = 5x$ *implies that* $P' = 5$. *Substituting* P *and* P' *into* (4.11), *we obtain* $Q = 5$. *Similarly plugging* P, Q *and* Q' *into* (4.12), $R = 0$. *Therefore we have* $P = 5x$, $Q = -5$ *and* $R = 0$ *which when substituted into the ODE*

$$y''' + Py'' + Qy' + Ry = f(x)$$

gives

$$y''' + 5xy'' - 5y' = f(x).$$

Then by using (3.4) *i.e.*

$$[u(y'' + by' + ay)]' = uf(x), \tag{4.13}$$

and choosing for example $b = b_1 = 5x$*, then from relation* (3.10) *solving for* a *we have as* $a = b^2 - bP + Q - b'$ *then plugging in* $b = 5x$*,* $P = 5x$ *and* $Q = -5$*, we have* $a = -10$*. Now using relation* (3.8) *i.e.*

$$\frac{u'}{u} = P - b \tag{4.14}$$

which implies that

$$\frac{u_1'}{u_1} = P - b_1$$

then we can solve for u_1 *by substituting* $P = 5x$ *and* $b_1 = 5x$*. Thus*

$$\frac{u_1'}{u_1} = 5x - 5x = 0$$

which means that u_1 *is a constant.*

Replacing $u = c, b = b_1$ *and* $a = a_1$ *in* (4.13)*, we have:*

$$[c(y'' + b_1y' + a_1y)]' = cf(x).$$

If we choose $f(x) = 0$ *then,*

$$[c(y'' + b_1y' + a_1y)]' = 0.$$

Integrating both sides of the above equation, we get

$$y'' + b_1y' + a_1y = c.$$

Substituting $b_1 = 5x$ *and* $a_1 = -10$ *we have*

$$y'' + 5xy' - 10y = c$$

and as $y = (x^2 + 1/5)$ *is a solution of this equation, it is possible to solve the problem.*

Next as $b_2 = -\frac{1}{x}$, we have

$$\frac{u'}{u} = P - b_2 = 5x + \frac{1}{x}$$

Hence solving for u we obtain $u = xe^{\frac{5}{2}x^2}$. Similarly, solving for $a_2 = b_2^2 - b_2P + Q - b_2'$ where $b_2 = -\frac{1}{x}$, $P = 5x$ and $Q = -5$, we obtain $a_2 = 0$. In this case our equation becomes

$$\left[xe^{\frac{5}{2}x^2}(y'' - \frac{1}{x}y')\right]' = xe^{\frac{5}{2}x^2}f(x). \tag{4.15}$$

If we choose $f(x) = 0$ then,

$$\left[xe^{\frac{5}{2}x^2}(y'' - \frac{1}{x}y')\right]' = 0. \tag{4.16}$$

Integrating both sides of the above equation, we get

$$xe^{\frac{5}{2}x^2}(y'' - \frac{1}{x}y') = c. \tag{4.17}$$

Thus it is possible to find y.

The next case is $b_3 = -\frac{1}{x}\left[1 + \frac{1}{x}\frac{e^{-\frac{5}{2}x^2}}{\int \frac{e^{-\frac{5}{2}x^2}}{x^2}}\right]$.

In all these case, b_1, b_2 and b_3 satisfy that (4.8). Also using the fact that $b_3 = -\frac{1}{x}\left[1 + \frac{1}{x}\frac{e^{-\frac{5}{2}x^2}}{\int \frac{e^{-\frac{5}{2}x^2}}{x^2}}\right]$, we have

$$\frac{u'}{u} = P - b_3 = 5x + \frac{1}{x}\left[1 + \frac{1}{x}\frac{e^{-\frac{5}{2}x^2}}{\int \frac{e^{-\frac{5}{2}x^2}}{x^2}}\right]$$

Also, solving for $a_3 = b_3^2 - b_3P + Q - b_3'$ where b_3 is defined as above, $P = 5x$ and $Q = -5$, we obtain $a_3 = 0$. It is true that $(ua)' = uR = 0$ since $a = 0$, $R = 0$, $u \neq 0$ and $u \neq c$ where c is a constant.

Next if on the other hand we have that as defined in (3.22),

$$u''' - u''P + u'(Q - 2P') + u(Q' - P'' - R) = 0. \tag{4.18}$$

with corresponding equation

$$u''' + 10xu'' + 25x^2u' + -25xu = 0. \tag{4.19}$$

with $u_1 = x$, $u_2 = e^{\frac{5}{2}x^2}$ *and* $u_3 = x\int \frac{e^{-\frac{5}{2}x^2}}{x^2}dx$, *then driving the correspondence:* $-P = 10x$, $Q - 2P' = 25x^2$ *and* $Q' - P'' - R = -25x$. *Solving these we have* $P = -10x$, $Q = 25x^2 - 20$ *and* $R = 75x$. *So these correspond to*

$$y''' - 10xy'' + (25x^2 - 20)y' + 75xy = 0.$$

As

$$\frac{u'}{u} = P - b = -10x - b$$

then knowing u *we have* b. *Therefore letting* $u_1 = x$, *we have*

$$\frac{u_1'}{u_1} = P - b_1$$

and so

$$\frac{u_1'}{u_1} = \frac{1}{x} = -10x - b_1.$$

This implies that $b_1 = -10x - \frac{1}{x}$. *Solving for* $a_1 = b_1^2 - b_1P + Q - b_1'$ *where* $P = -10x$, $Q = 25x^2 - 20$ *and* $b_1 = -10x - \frac{1}{x}$, *we have* $a_1 = 25x^2$. *So we see that* $(ua)' = uR = 75x^2$.

With these values considering the original equation,

$$y''' - 10xy'' + (25x^2 - 20)y' + 75xy = f(x)$$

or

$$[u(y'' + by' + ay)]' = uf(x),$$

if $f(x) = 0$ *we have with these values of* $u_1 = x$, $a_1 = 25x^2$ *and* $b_1 = -10x - \frac{1}{x}$ *that*

$$\left[x(y'' + (-10x - \frac{1}{x})y' + 25x^2y)\right]' = 0.$$

On the other hand, if $u_2 = e^{-\frac{5}{2}x^2}$, *then*

$$\frac{u_2'}{u_2} = P - b_2$$

and so plugging in u_2 and P and solving for b_2 , we obtain $b_2 = -5x$. Also solving for $a_2 = b_2^2 - b_2 P + Q - b_2'$ where $P = -10x$, $Q = 25x^2 - 20$ and $b_2 = -5x$, we have $a_2 = -15$. Therefore substituting these values into

$$[u_2(y'' + b_2 y' + a_2 y)]' = u_2 f(x),$$

we obtain

$$\left[e^{-\frac{5}{2}x^2}(y'' - 5xy' - 15y)\right]' = e^{\frac{5}{2}x^2} f(x),$$

and if $f(x) = 0$ and we integrate both sides of the above equation, we obtain:

$$y'' - 5xy' - 15y = ce^{\frac{5}{2}x^2}$$

Since the solution for $y'' - 5xy' - 15y = 0$ are $y_1 = (5x^2+1)e^{\frac{5}{2}x^2}$ and $y_2 = (5x^2+1)e^{\frac{5}{2}x^2} \int \frac{e^{-\frac{5}{2}x^2}}{(5x^2+1)^2} dx$ then we can solve this using:

$$(u(y' + by))' = uce^{\frac{5}{2}x^2}$$

where we use the theory of integrating factor for the second degree where $\frac{u'}{u} = P - b$ and $b = -\frac{y'}{y}$. Therefore

$$b = -\frac{y'}{y} = -\frac{\left[(5x^2+1)e^{\frac{5}{2}x^2}\right]'}{(5x^2+1)e^{\frac{5}{2}x^2}}.$$

Simplifying the above relation, we obtain

$$b = -5x - \frac{10x}{5x^2+1}.$$

Substituting this value of b into the relation $\frac{u'}{u} = P - b$ with $P = -5x$ we have

$$\frac{u'}{u} = -5x + 5x + \frac{10x}{5x^2+1} = \frac{10x}{5x^2+1}$$

Solving for u, we obtain $u = 5x^2 + 1$. This implies that

$$(5x^2+1)\left[y' + \left(-5x - \frac{10x}{5x^2+1}\right)y\right]' = (5x^2+1)ce^{\frac{5}{2}x^2}$$

and so integrating both sides of the above equation, we obtain

$$(5x^2+1)\left[y' + \left(-5x - \frac{10x}{5x^2+1}\right)y\right] = c\int(5x^2+1)e^{\frac{5}{2}x^2}dx = cxe^{\frac{5}{2}x^2} + k.$$

Thus
$$y' + \left(-5x - \frac{10x}{5x^2+1}\right) y = \frac{cxe^{\frac{5}{2}x^2}}{5x^2+1} + \frac{k}{5x^2+1}.$$
Now integrating factor of the first degree implies that
$$(uy)' = u\left[\frac{cxe^{\frac{5}{2}x^2}}{5x^2+1} + \frac{k}{5x^2+1}\right]$$
where $u = e^{\int -5x - \frac{10x}{5x^2+1}dx} = \frac{e^{-\frac{5}{2}x^2}}{5x^2+1}$. *Therefore*
$$\left(\frac{e^{-\frac{5}{2}x^2}}{5x^2+1}y\right)' = \frac{e^{-\frac{5}{2}x^2}}{5x^2+1}\left[\frac{cxe^{\frac{5}{2}x^2}}{5x^2+1} + \frac{k}{5x^2+1}\right]$$
Integrating both sides of the above equation, we get:
$$\frac{e^{-\frac{5}{2}x^2}}{5x^2+1}y = \int \frac{cx}{(5x^2+1)^2}dx + \int \frac{ke^{-\frac{5}{2}x^2}}{(5x^2+1)^2}dx + K_1$$
Solving for y *we obtain:*
$$y = (5x^2+1)e^{\frac{5}{2}x^2}\int \frac{cx}{(5x^2+1)^2}dx + (5x^2+1)e^{\frac{5}{2}x^2}\int \frac{ke^{-\frac{5}{2}x^2}}{(5x^2+1)^2}dx + K_1(5x^2+1)e^{\frac{5}{2}x^2}$$
Simplifying further, we obtain:
$$y = (5x^2+1)e^{\frac{5}{2}x^2}c\frac{-1}{10(5x^2+1)} + k(5x^2+1)e^{\frac{5}{2}x^2}\int \frac{e^{-\frac{5}{2}x^2}}{(5x^2+1)^2}dx + K_1(5x^2+1)e^{\frac{5}{2}x^2}$$
So we are able to find the solution for the ODE
$$y''' - 10xy'' + (25x^2-20)y' + 75xy = f(x)$$
If we choose $f(x) = 0$, *then* $y_1 = e^{\frac{5}{2}x^2}$, $y_2 = (5x^2+1)e^{\frac{5}{2}x^2}\int \frac{e^{-\frac{5}{2}x^2}}{(5x^2+1)^2}dx$ *and* $y_3 = (5x^2+1)e^{\frac{5}{2}x^2}$. *The interesting fact is that starting with the original differential equation,*
$$y''' + 10xy'' + 25x^2y' - 25xy = 0.$$
with solutions $y_1 = x$, $y_2 = e^{-\frac{5}{2}x^2}$ *and* $y_3 = x\int \frac{e^{-\frac{5}{2}x^2}}{x^2}dx$ *we found solutions* $y_1 = e^{\frac{5}{2}x^2}$, $y_2 = (5x^2+1)e^{\frac{5}{2}x^2}\int \frac{e^{-\frac{5}{2}x^2}}{(5x^2+1)^2}dx$ *and* $y_3 = (5x^2+1)e^{\frac{5}{2}x^2}$ *for the ODE*
$$y''' - 10xy'' + (25x^2-20)y' + 75xy = 0.$$

Example 4.3. *Consider a third order ODE,*

$$y''' - 12y'' + 48y' - 64y = 12 - 32e^{-8x} + 2e^{4x}. \tag{4.20}$$

Comparing (4.20) *with equation* (3.1) *we have* $P(x) = -12$, $Q(x) = 48$, $R(x) = -64$ *and* $f(x) = 12 - 32e^{-8x} + 2e^{4x}$. *This equation can be solved to obtain the homogeneous solution and particular solution as,*

$$y(x) = C_1 e^{4x} + C_2 x e^{4x} + C_3 x^2 e^{4x} - \frac{3}{16} + \frac{1}{54} e^{-8x} + \frac{1}{3} x^3 e^{4x} \tag{4.21}$$

where $y_h(x) = C_1 e^{4x} + C_2 x e^{4x} + C_3 x^2 e^{4x}$ *represents the homogeneous solution and* $y_p(t) = -\frac{3}{16} + \frac{1}{54} e^{-8x} + \frac{1}{3} x^3 e^{4x}$ *represents the particular solution.*

Now substituting $P(x) = -12$, $Q(x) = 48$ *and* $R(x) = -64$ *into equation* (3.20) *yields,*

$$y''' - 24y'' + 192y' - 512y = 0. \tag{4.22}$$

Two solutions of the above equation are $y_1 = e^{8x}$ *and* $y_2 = xe^{8x}$, *then using the relation* $b = \frac{-y'}{y}$ *we find two solutions of equation* (3.14) *which after substituting the values of* $P(x)$,$Q(x)$ *and* $R(x)$ *yields*

$$b'' = 65 - 576 + -192b + 3bb' + 24b' - 24b^2 - b^3. \tag{4.23}$$

Using the expression for b *gives* $b = \frac{-y_1'}{y_1} = -8$ *and* $b = \frac{-y_2'}{y_2} = \frac{-1-8x}{x}$ *as two solutions to equation* (4.23)*.This is verified by substituting both solutions into equation* (4.23).

Now since we have a solution for b *we can find* a *by using the relation* $a = Q - b' - b(P - b)$.
Suppose we use the first solution $b = -8$, *then we have* $a = 16$, *then substituting* a *into* (3.12) *we can find the solution for* u *as* e^{-4x}.

With u *known as well as* a *and* b *and we substituting them into equation* (3.4) *yields,*

$$\left(e^{-4x}(y'' - 8y' + 16y)\right) = e^{-4x}(12 - 32e^{-8x} + 2e^{4x}), \tag{4.24}$$

which is simplified into

$$y'' - 8y' + 16y = -3 + \frac{8}{3}e^{-8x} + \frac{1}{3}x^3e^{4x}. \tag{4.25}$$

The solution of equation (4.25) *can be found to be,*

$$y(x) = Ae^{4x} + Bxe^{4x} - \frac{3}{16} + \frac{1}{54}e^{-8x} + \frac{1}{3}x^3e^{4x} \tag{4.26}$$

where $y_h(x) = Ae^{4x} + Bxe^{4x}$ *represents the homogeneous solution and* $y_p(t) = -\frac{3}{16} + \frac{1}{54}e^{-8x} + \frac{1}{3}x^3e^{4x}$ *represents the particular solution. By comparison, the particular solution found for equation* (4.25) *is the same as the particular solution of equation* (4.20)*, i.e. the original ODE. We also see that the homogeneous solution of equation* (4.25) *represents 2 of the solutions of the homogeneous part of equation* (4.20)*.*

Next, Suppose we had a solution of equation (4.20) *given as* $y = Ce^{4x}$*, then from* (3.19)*, after making the necessary substitutions we get,*

$$b'' = -64 - 12b^2 + 3bb' - b^3 - 48b + 12b'. \tag{4.27}$$

Using the expression $b = \frac{-y'}{y}$ *we get the solution* $b = -4$ *which is a solution of* (4.27)*. It can be shown also that using* $y = Axe^{4x}$ *gives another solution to equation* (4.23) *as* $b = -\left(\frac{1+4x}{x}\right)$*.*

Finally, knowing that $u(x) = e^{-4x}$ *we can find* b *from the relation* $\frac{u'}{u} = P - b$ *yielding* $b = -8$ *which we have shown earlier to be a solution to* (4.23)

Example 4.4. *Consider the third order ODE*

$$u''' + x^2u'' + 6xu' + 6u = 0. \tag{4.28}$$

One solution for (4.28) *is* $u_1 = x^2e^{-x^3/3}$ *and another solution is* $u_2 = x^2e^{-x^3/3}\int \frac{e^{x^3/3}}{x^3}dx$*.*

Comparing (4.28) *to the integrating factor equation* (3.22)*, we observe that* $-P = x^2$*,* $Q - 2P' = 6x$ *and* $Q' - P'' - R = 6$*. Solving for* P, P', P'', Q, Q' *and* R*, the corresponding equation for* (3.1) *is*

$$y''' - x^2y'' + 2xy' - 2y = f(x).$$

Since $\frac{u'}{u} = P - b$ *from* (3.8)*, we can solve for* b *using* $u = e^{-x^3/3}x^2$*. Thus,* $\frac{u'}{u} = \frac{2}{x} - x^2$ *and as* $\frac{u'}{u} = P - b = -x^2 - b$ *solving for* b *we get* $b = -\frac{2}{x}$*. Now,* *from (3.10), since $a = Q - b' - b(P - b)$, replacing b, P, Q and b', we get $a = \frac{2}{x^2}$. Thus, replacing u, b and a in (3.4) and integrating both sides we get:*

$$e^{-x^3/3}x^2\left(y'' - \frac{2}{x}y' + \frac{2}{x^2}y\right) = \int e^{-x^3/3}x^2 f(x)dx.$$

Suppose $f(x) = c$, then

$$e^{-x^3/3}x^2\left(y'' - \frac{2}{x}y' + \frac{2}{x^2}y\right) = -ce^{-x^3/3} + k$$

and we have that:

$$y'' - \frac{2}{x}y' + \frac{2}{x^2}y = -\frac{c}{x^2} + k\frac{e^{x^3/3}}{x^2}.$$

To find a solution for this last equation, we use Euler for the homogeneous equation $y'' - \frac{2}{x}y' + \frac{2}{x^2}y = 0$ or

$$x^2y'' - 2xy' + 2y = 0. \tag{4.29}$$

Setting $y = x^r$, we get the fundamental solutions to be $y_1 = x^2$ and $y_2 = x$. Hence, the general solution for the homogeneous part is $y_h = c_1x^2 + c_2x$. To find a particular solution, we implement again the technique used in the previous example.

Consider

$$y'' - \frac{2}{x}y' + \frac{2}{x^2}y = g(x) \tag{4.30}$$

with $g(x) = -\frac{c}{x^2} + k\frac{e^{x^3/3}}{x^2}$. Using one of the fundamental solutions for example $y = x$, since $b = -\frac{y'}{y}$, then, from Equation (2.3) with $g(x)$ instead of $R(x)$, we get:

$$(u(y' + by))' = ug(x). \tag{4.31}$$

We recall that $\frac{u'}{u} = P - b$, where in this case from (4.30), $P = -\frac{2}{x}$ and $b = -\frac{y'}{y} = -\frac{1}{x}$. Thus, $\frac{u'}{u} = -\frac{1}{x}$, hence $u = x^{-1}$. Thus, replacing u and b into (4.31), we have:

$$\left(x^{-1}\left(y' + (-\frac{1}{x})y\right)\right)' = \frac{1}{x}g(x), \text{ then } \frac{1}{x}\left(y' - \frac{1}{x}y\right) = \int \frac{1}{x}g(x)dx + \tilde{k}$$

and, for this last equation, we obtain the solution by using the first degree order integrating factor technique.

Now going back to the corresponding equation for (4.28)*, recall that we found* $b = \frac{-2}{x}$*,* $a = \frac{2}{x^2}$*,* $\mu = e^{\frac{-x^2}{3}} x^2$*,* $P = -x^2$*,* $Q = 2x$ *and* $R = -2$*. Inserting the expressions for* a*,* a'*,* a''*,* Q*,* P*,* R *in*

$$a''a = a'(aP - 3R) + 2(a')^2 + a(R' - PR) + a^2(Q - P') + R^2 - a^3$$

We obtain $a''a = \frac{24}{x^6}$ *which is the correct result.*

5 Conclusion

In this work, we presented analytic methodologies for solving some particular classes of third order ODEs. The methodologies presented may serve as a reference for solving other higher order ODEs.

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