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Geometric Power Series In Action

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Abstract

The Geometric Power Series is a series of the form $\sum_{k=0}^{\infty} ax^k$. The purpose of this study is to investigate some important applications of the geometric power series $\sum_{k=0}^{\infty} x^k$ in |x|<1. Geometric series appear everywhere, and here we try to do a good job in showing the different aspects of it. The geometric series is one of the easiest examples of power of a power series. It is also one of the most useful and applicable power series. It is commonly used in differential equations and engineering. We will show that this power series has surprisingly many applications by giving 9 different impressive applications.

Introduction and History

Power series representation of functions has played an important role in the development of calculus. In fact, much of Newton's work with differentiation and integration was in the context of power series-especially his work with complicated algebraic functions and transcendental functions. Euler, Lagrange, Leibniz, and the Bernoulli's all used power series exclusively in calculus.

The representation of functions by power series is one of the most useful of mathematical techniques in a wide variety of situations.

Sometimes we start from a function that is defined for us in some manner not employing series, and seek to expand the function in a series. At other times we may form a power series, or have one represented to us, and then undertake to use this function in some way. In either of these situations we need to know something of what properties a function has if it is defined by a power series.

Let f(x) be the function represented by the series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

Then f(x) is called a power series function.

More generally, if f(x) is represented by the series

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

Then we call f(x) a power series centered at x=c. The domain of f(x) is called the Interval of Convergence and half the length of the domain is call the Radius of Convergence.

Among all power series the one which plays the most important role in applications is the Geometric Power Series which is given as

$$f(x) = \sum_{n=0}^{\infty} ax^n$$

The radius of convergence of this power series is 1.

Background Materials

1. $\sum_{n=k}^{\infty} a_n = \sum_{k=0}^{\infty} a_{(n+k)}$
2. $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
3. Once a function is given as a power series, it is continuous in a set wherever it converges and differentiable on the interior of this set.
4. The power series can be differentiated and integrated quite easily, by treating every term separately:

1. $f'(x) = \sum_{n=1}^{\infty} a_n n(x-c)^{n-1}$
2. $\int f(x)dx = \sum_{n=0}^{\infty} \frac{a_n n(x-c)^{n-1}}{n+1} + C$

5. The radius of convergence of a power series is a non-negative quantity – either a real number or +,∞ - that represents a range (within the radius) in which the function will converge.

   For a power series f defined as:

$$f(z) = \sum_{n=0}^{\infty} f_n = \sum_{n=0}^{\infty} c_n(z-a)^n,$$

   where

   a is a constant, the center of the circle of convergence,
   c_n is the *n*th complex coefficient (note that real numbers are a very common special case of complex numbers), z is a variable
   f_n is the *n*th term of the series

6. The radius of convergence r is a nonnegative real number of ∞ , such that the series converges if

$$|z-a| < r$$

   and diverges if

$$|z-a| > r.$$

   In other words, the series converges if z is close enough to the center and diverges if it is too far away. The radius of convergence specifies how close is close enough. The radius of convergence is infinite if the series converges for all complex numbers z.

7. We commonly use the ratio and root tests to find the interval where the series is absolutely convergent.

8. To obtain the derivative or the integral of a power series function, we can pass the derivative or integral through Σ within its radius of convergence.

The Main Results

Lemma 1: If -1<x<1 and $S_n = \sum_{k=0}^{n} x^k$ then $S_n = \frac{1-x^{n+1}}{1-x}$

Proof: If $S_n = 1 + x + x^2 + \ldots + x^n$ then $xS_n = x + x^2 + x^3 + x^n + x^{n+1}$

and it follows that ,

$$S_n - xS_n = 1 - x^{n+1}$$

$$S_n(1-x) = \frac{1-x^{n+1}}{1-x}$$

$$S_n = \frac{1-x^{n+1}}{1-x}$$

Remark I: If x>1, then $S_n = \frac{x^{n+1}-1}{x-1}$

Remark II: If $|x| < 1$, the $\lim_{n\to\infty} x^n = 0$

Theorem 1: If $|x| < 1$, then $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$

Proof: The result easily follows by Lemma I and Remarks I and II as

$$\lim_{n\to\infty}\left(\sum_{k=0}^{n}x^k\right)=\lim_{n\to\infty}\left(\frac{1-x^{n+1}}{1-x}\right)=\frac{1}{1-x}$$ implies that $$\sum_{k=0}^{\infty}x^k=\frac{1}{1-x}$$

<u>Some Important Applications Of Geometric Power series</u>

***Application 1**:*

Find the exact sum of : $\sum_{k=2}^{\infty}\left(\frac{3^k+4^k}{6^k}\right)$

<u>Solution</u>:

$$\sum_{k=2}^{\infty}\left(\frac{3^k+4^k}{6^k}\right)$$

$$= \sum_{k=2}^{\infty}\left(\frac{3^k}{6^k}+\frac{4^k}{6^k}\right)$$

$$= \sum_{k=2}^{\infty}\left(\frac{1}{2}\right)^k+\sum_{k=2}^{\infty}\left(\frac{2}{3}\right)^k$$

$$= \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^{k+2}+\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^{k+2}$$

$$= \sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^2+\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k\left(\frac{2}{3}\right)^2$$

$$= \frac{1}{4}\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k+\frac{4}{9}\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k$$

$$= \frac{1}{4}\left(\frac{1}{1-\frac{1}{2}}\right)+\frac{4}{9}\left(\frac{1}{1-\frac{2}{3}}\right)$$

$$= \frac{1}{4}(2) + \frac{4}{9}(3) = \frac{11}{6}$$

Application 2:

Simplify: $\frac{66+48+24+12+6+.....}{54+36+24+.....}$

Solution:

$$= \frac{66+48+24+12+6+....}{54+36+24+....}$$

$$= \frac{66+48\left(1+\frac{1}{2}+\frac{1}{4}...\right)}{54\left(1+\frac{2}{3}+\frac{4}{9}+....\right)}$$

$$= \frac{66+48\left(\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k\right)}{54\left(\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k\right)}$$

$$= \frac{66+48\left(\frac{1}{1-\frac{1}{2}}\right)}{54\left(\frac{1}{1-\frac{2}{3}}\right)}$$

$$= \frac{66+48(2)}{54(3)}$$

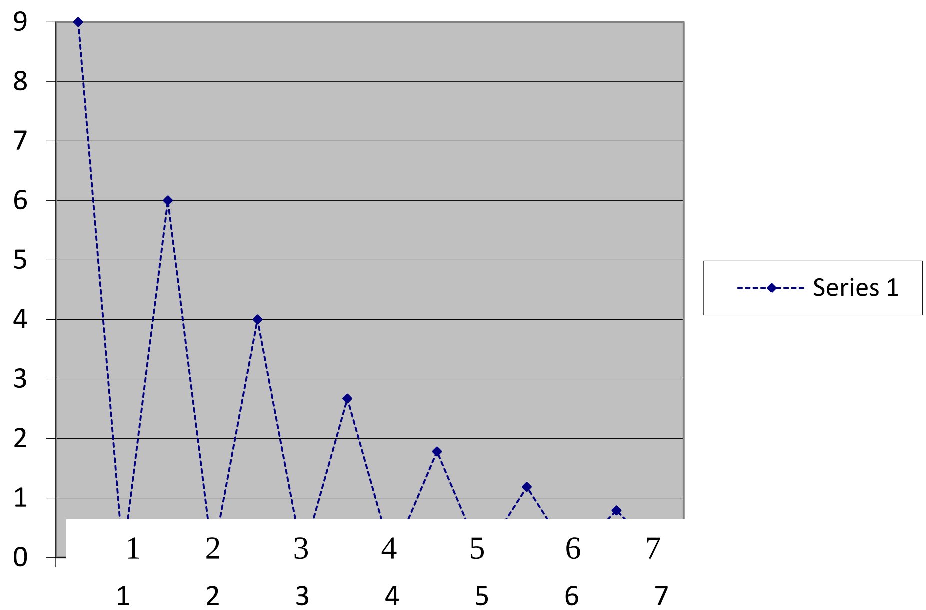
$$= \frac{66+96}{162}$$

$$= \frac{162}{162}$$

$$= 1$$

Application 3: Application to Physics (Bouncing Ball Problem)

A ball is dropped from a height of 9 feet and begins bouncing, as shown below. The height of each bounce is two-thirds the height of the previous bounce. Find the total vertical distance traveled by the ball using Geometric Power Series property.



Solution

When the ball hits the ground for the first time, it has traveled a distance of $D_1 = 9$ feet. For subsequent bounces, let D_i be the distance traveled up and down. For example, D_2 and D_3 are as follows.

$$D_2 = 9(\frac{2}{3}) + 9(\frac{2}{3}) = 18(\frac{2}{3})$$

$$D_3 = 9(\frac{2}{3})(\frac{2}{3}) + 9(\frac{2}{3})(\frac{2}{3}) = 18(\frac{2}{3})^2$$

By continuing this process it can be determined that the total vertical distance is

$$D = 9 + 18(\frac{2}{3}) + 18(\frac{2}{3})^2 + 18(\frac{2}{3})^3 + ...$$

$$= 9 + 18\sum_{n=0}^{\infty}(\frac{2}{3})^{n+1}$$

$$= 9 + 18(\frac{2}{3})\sum_{n=0}^{\infty}(\frac{2}{3})^{n}$$

$$= 9 + 12\left(\frac{1}{1-\frac{2}{3}}\right)$$

$$= 9 + 12(3)$$
$$= 45 Feet$$

Application 4: The sides of a square are each 8 cm long. A second square is inscribed by joining the midpoints of the sides, successively. In the second square we repeat the process, inscribing a third square. If the process is continued indefinitely, what is the sum of all of the areas of the all the squares?

Solution: Observe that the sum of the areas has the following pattern.

64+32+16+8+… and the sum of all the areas will be:

64(1+$\frac{1}{2}+\frac{1}{4}+\ldots$) = 64(2)= 128 cm^2.

Applications5.: Find the Power Series Expansion of f(x)= $\frac{1}{3-x}$

Solution

f(x)= $\frac{1}{3-x}$

$$= \frac{1}{3-x}$$

$$= \frac{1}{3(1-x/3)}$$

$$= \frac{1}{3}\sum_{k=0}^{\infty}\left(\frac{x}{3}\right)^k$$

$$= \sum_{k=0}^{\infty}\frac{x^k}{3^{k+1}} \qquad (|x|<3)$$

Geometric Power Series and Derivatives

Observe that within the radius of convergence:

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n \Rightarrow$$

$$f'(x) = \sum_{n=1}^{\infty} a_n n(x-c)^{n-1} =$$

$$= \sum_{n=0}^{\infty} a_{n+1}(n+1)(x-c)^n$$

Some Application Problems Using The Above Derivatives Rule

Application 6*:* Find the exact sum of $\sum_{k=2}^{\infty}(k-1)(k)\left(\frac{1}{2}\right)^{k+2}$

Solution:

Note that:

$$\sum_{k=2}^{\infty}(k-1)(k)\, x^{k+2} =$$

$$\sum_{k=0}^{\infty}(k+2)(k+1)x^k$$

and

$$\sum_{k=0}^{\infty}(k+2)(k+1)\left(\frac{1}{2}\right)^{k+4} =$$

$$\frac{1}{16}\sum_{k=0}^{\infty}(k+2)(k+1)\left(\frac{1}{2}\right)^{k} = \frac{1}{16}\left(\frac{2}{\left(1-\frac{1}{2}\right)^{3}}\right)$$

$$\Rightarrow \frac{1}{16}\sum_{k=0}^{\infty}(k+2)(k+1)\left(\frac{1}{2}\right)^{k} = \frac{1}{16}(16) = 1$$

Applications 7:

Find the exact sum of: $\sum_{k=0}^{\infty}(-1)^{k}(k+1)\left(2^{-2k}+3^{-k}\right)$

<u>Solution:</u>

$$\sum_{k=0}^{\infty}(-1)^{k}(k+1)\left(2^{-2k}+3^{-k}\right)$$

=

$$\sum_{k=0}^{\infty}(-1)^{k}(k+1)(2)^{-2k} + \sum_{k=0}^{\infty}(-1)^{k}(k+1)(3)^{-k}$$

$$= \sum_{k=0}^{\infty}(k+1)\left(\frac{-1}{4}\right)^{k} + \sum_{k=0}^{\infty}(k+1)\left(-\frac{1}{3}\right)^{k}$$

$$= \sum(k+1)\left(\frac{-1}{4}\right)^{k} + \sum_{k=0}^{\infty}(k+1)\left(\frac{-1}{3}\right)^{k}$$

Note that:

$\sum_{k=0}^{\infty}(k+1)x^{k} = \frac{1}{(1-x)^{2}}$ and hence,

$$\sum_{k=0}^{\infty}(k+1)\left(\frac{-1}{4}\right)^k+\sum_{k=0}^{\infty}(k+1)\left(\frac{-1}{3}\right)^k=\frac{1}{\left(1+\frac{1}{4}\right)^2}+\frac{1}{\left(1+\frac{1}{3}\right)^2}$$

$$\Rightarrow\sum_{k=0}^{\infty}(k+1)\left(\frac{-1}{4}\right)+\sum_{k=0}^{\infty}(k+1)\left(\frac{-1}{3}\right)=\frac{1}{\left(\frac{5}{4}\right)^2}+\frac{1}{\left(\frac{4}{3}\right)^2}=\frac{1}{\frac{25}{16}}+\frac{1}{\frac{16}{9}}$$

$$\Rightarrow\sum_{k=0}^{\infty}(k+1)\left(\frac{-1}{4}\right)+\sum_{k=0}^{\infty}(k+1)\left(\frac{-1}{3}\right)^k=\frac{481}{400}$$

Application 8:

Find the power series expansion of f(x)= $\frac{1}{x^3}$

Solution:

Note that,

$$\frac{1}{x^3}=\frac{1}{(1-(1-x))^3}$$

$$=\frac{1}{2}\sum_{k=0}^{\infty}(k+2)(k+1)(1-x)^3$$

Geometric Series and Integration

Remark III : If $\sum_{k=0}^{\infty}x^k=\frac{1}{1-x}$ and $|x|<1$, then

$$\int\left(\sum_{k=0}^{\infty}x^k\right)dx=\int\frac{dx}{1-x}\Rightarrow\sum_{k=0}^{\infty}\frac{x^{k+1}}{k+1}=\ln(1-x)+c$$

Note if x=0, we have c=0 and hence $\sum_{k=0}^{\infty}\frac{x^{k+1}}{k+1}$ = -ln(1-x)

Application 9:

Find the exact sum of: $\sum_{k=0}^{\infty}\frac{1}{k+1}\left(\frac{1}{2}\right)^k$.

Solution:

Note: $\sum_{k=0}^{\infty}\frac{1}{k+1}x^{k+1}=-\ln(1-x), -1<x<1$ and hence,

$$\sum_{k=0}^{\infty}\frac{1}{k+1}\left(\frac{1}{2}\right)^{k}=2\sum_{k=0}^{\infty}\frac{1}{k+1}\left(\frac{1}{2}\right)^{k+1}=-2\ln\left(\frac{1}{2}\right)=2\ln 2$$

Application 10:

Find the exact sum of: $\sum_{k=0}^{\infty}(-1)^{k}\frac{1}{2k+1}\left(\frac{1}{\sqrt{3}}\right)^{2k+1}$.

<u>Solution</u>:

Note that:

$$\frac{1}{1+x}=\sum_{k=0}^{\infty}(-1)^{k}x^{k}$$

$$\Rightarrow \frac{1}{1+x^{2}}=\sum_{k=0}^{\infty}(-1)^{k}x^{2k}$$

$$\Rightarrow \int\frac{dx}{1+x^{2}}=\int\sum_{k=0}^{\infty}(-1)^{k}x^{2k}dx$$

$$\Rightarrow Tan^{-1}(x)+c=\sum_{k=0}^{\infty}(-1)^{k}\frac{x^{2k+1}}{2k+1}$$

$$\Rightarrow Tan^{-1}(x)=\sum_{k=0}^{\infty}(-1)^{k}\frac{x^{2k+1}}{2k+1}$$ (since x=0 $\Rightarrow c=0$)

$$\Rightarrow Tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=\sum_{k=0}^{\infty}(-1)^{k}\frac{1}{(2k+1)}\left(\frac{1}{\sqrt{3}}\right)^{2k+1}$$

$$\frac{\pi}{6}=\sum_{k=0}^{\infty}(-1)^{k}\frac{1}{2k+1}\left(\frac{1}{\sqrt{3}}\right)^{2k+1}$$

Conclusion:

We believe that the concept studied investigated in this paper will play a big role in the future teaching and learning of the Geometric Power Series in calculus class and other related.

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References:

1. Gillman, Leonard and Robert H. McDowell. Calculus.1st Ed. University of Texas Austin. W.W. Norton & Company, Inc. New York. 1973.
2. Taylor, Angus E. Advanced Calculus. University of California. Blaisdell Publishing Company. Los Angeles. 1955.
3. Shakera Spencer, Alicia Ware, Mulatu Lemma, and The Geometric Power Series are everywhere.