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Abstract. In this paper, we study a Vandermonde type of determinants. We prove an explicit formula for this type of determinants and the property of anti-symmetry. The results have applications in algebraic geometry and Arithmetic geometry, but it could potentially have applications in other fields where the Vandermonde type of determinants are encountered.

Key words. Vandermonde matrix, determinant, polynomial interpolation

AMS subject classifications. 15A15, 14A10.

1. Statements. In numerical analysis, the main tool for polynomial interpolation is the Vandermonde matrix ([1]). The Vandermonde determinant also plays a central role in the Frobenius formula, which gives the character of conjugacy classes of representations of the symmetric group ([2]). In the polynomial interpolation, the existence of the Lagrange interpolation polynomial that is interpolated into the given points is automatic because the number of given points is the same as the number of coefficients in the Lagrange interpolation polynomial. In this paper, we try to answer the question: when can a polynomial of fixed degree be interpolated into the given points if the number of points is larger than the number of coefficients of the interpolation polynomial? The answer lies in our extension of "Vandermonde matrix". We call it the "augmented Vandermonde matrix". See Example (3.1) below.

This problem arises from algebraic geometry in the case when the number of constraints is larger than the number of variables. See our example (3.2) in the last section. We hope this will give more applications to different situation when the existence question arises.

Let's first define the "augmented Vandermonde matrix". Let $\mathbf{K}$ be an algebraically closed field (some formulas work in any field). Let $h(t)$ be a monic polynomial of degree $n+2$ in $\mathbf{K}(t)$ without multiple roots for some non-negative integer n, where a monic polynomial is a polynomial with leading coefficient 1. We'll denote $h(t)$ by h. Then $h(t)$ can be factorized as

$$h(t) = \prod_{i=1}^{n} (t - \theta_i), \tag{1.1}$$

where $\theta_i \in \mathbf{K}$. Let R be any $\mathbf{K}$ valued function on $\mathbf{K}$.

We define the normalized determinant of an "augmented Vandermonde matrix"

$$V(h,R)=\frac{1}{v_h}\begin{vmatrix} 1 & \theta_1 & (\theta_1)^2 & \cdots & (\theta_1)^n & R(\theta_1) \\ 1 & \theta_2 & (\theta_2)^2 & \cdots & (\theta_2)^n & R(\theta_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \theta_{n+2} & (\theta_{n+2})^2 & \cdots & (\theta_{n+2})^n & R(\theta_{n+2}) \end{vmatrix} \tag{1.2}$$

where

$$v_h=\prod_{i>j}(\theta_i-\theta_j) \tag{1.3}$$

is the determinant of the Vandermonde matrix

$$\begin{vmatrix} 1 & \theta_1 & (\theta_1)^2 & \cdots & (\theta_1)^n & (\theta_1)^{n+1} \\ 1 & \theta_2 & (\theta_2)^2 & \cdots & (\theta_2)^n & (\theta_2)^{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \theta_{n+2} & (\theta_{n+2})^2 & \cdots & (\theta_{n+2})^n & (\theta_{n+2})^{n+1} \end{vmatrix} \tag{1.4}$$

Notice that $V(h,R)$ is independent of the order of $\theta_1,\cdots,\theta_{n+2}$ because of the normalization denominator v_h.

THEOREM 1.1. *(a) Assume the R above is a rational function*

$$R=R_f=\frac{f(t)}{t-a}, \tag{1.5}$$

where $a\in\mathbf{K}, h(a)\neq 0$, and $f=f(t)\in\mathbf{K}(t)$ is a polynomial of degree $\leq n+1$ in t. Assume $h(t)$ has no multiple roots. Then the normalized determinant

$$V(h,R_f)=-\frac{f(a)}{h(a)}. \tag{1.6}$$

Furthermore if the R is a rational function $R=\frac{f}{g}$, $deg(f)-deg(g)\leq n$ that satisfies

(1) $g(t),h(t)$ do not have a common root, and g is a monic polynomial,

(2) both $g(t),h(t)$ do not have multiple roots. Then

$$V(h,\frac{f}{g})=-\sum_{\{w:g(w)=0\}}\frac{f(w)}{g'(w)h(w)} \tag{1.7}$$

where $g'(w)=\frac{dg(w)}{dt}$ is the derivative at w.

(b) (Anti-symmetry). Assume everything in part (a). Then there is an anti-symmetry between h and g,

$$V(h,\frac{f}{g})=-V(g,\frac{f}{h}) \tag{1.8}$$

An immediate consequence is that:

COROLLARY 1.2. *If $V(h,R_f)=0$ for some $h(t)$ with no multiple roots, then $f(a)=0$.*

Proof. : This is the direct consequence of (1.3). □

2. The proofs. *Proof.* of the theorem 1.1
(a). Use the long division, we obtain

$$\frac{f}{t-a} = p(t) + \frac{f(a)}{t-a}, \tag{2.1}$$

where $deg(p(t)) = n$. Notice

$$V(h, p(t)) = 0, \tag{2.2}$$

because the last column is a linear combination of the others. Hence

$$V(h, \frac{f}{t-a}) = f(a)V(h, \frac{1}{t-a}). \tag{2.3}$$

Now we change the variables to let

$$\tilde{\theta}_i = \theta_i - a. \tag{2.4}$$

Then

$$V(h, \frac{1}{t-a}) = \frac{1}{v_h} \begin{vmatrix} 1 & \tilde{\theta}_1 + a & (\tilde{\theta}_1 + a)^2 & \cdots & (\tilde{\theta}_1 + a)^n & (\tilde{\theta}_1)^{-1} \\ 1 & \tilde{\theta}_2 + a & (\tilde{\theta}_2 + a)^2 & \cdots & (\tilde{\theta}_2 + a)^n & (\tilde{\theta}_2)^{-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \tilde{\theta}_{n+2} + a & (\tilde{\theta}_{n+2} + a)^2 & \cdots & (\tilde{\theta}_{n+2} + a)^n & (\tilde{\theta}_{n+2})^{-1} \end{vmatrix}$$

By the linearity of the determinant on each column, we obtain

$$(2.5) V(h, \frac{1}{t-a}) = \frac{1}{v_h} \begin{vmatrix} 1 & \tilde{\theta}_1 + a & \cdots & (\tilde{\theta}_1 + a)^n & (\tilde{\theta}_1)^{-1} \\ 1 & \tilde{\theta}_2 + a & \cdots & (\tilde{\theta}_2 + a)^n & (\tilde{\theta}_2)^{-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \tilde{\theta}_{n+2} + a & \cdots & (\tilde{\theta}_{n+2} + a)^n & (\tilde{\theta}_{n+2})^{-1} \end{vmatrix}$$

$$(2.6) = \frac{1}{v_h} \begin{vmatrix} 1 & \tilde{\theta}_1 & \cdots & (\tilde{\theta}_1)^n & (\tilde{\theta}_1)^{-1} \\ 1 & \tilde{\theta}_2 & \cdots & (\tilde{\theta}_2)^n & (\tilde{\theta}_2)^{-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \tilde{\theta}_{n+2} & \cdots & (\tilde{\theta}_{n+2})^n & (\tilde{\theta}_{n+2})^{-1} \end{vmatrix}$$

$$(2.7) + \frac{1}{v_h} \sum_{(i_1 \cdots i_n)} c_{i_1 \cdots i_n}(a) \begin{vmatrix} 1 & \tilde{\theta}_1^{i_1} & (\tilde{\theta}_1)^{i_2} & \cdots & (\tilde{\theta}_1)^{i_n} & (\tilde{\theta}_1)^{-1} \\ 1 & \tilde{\theta}_2^{i_1} & (\tilde{\theta}_2)^{i_2} & \cdots & (\tilde{\theta}_2)^{i_n} & (\tilde{\theta}_2)^{-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \tilde{\theta}_{n+2}^{i_1} & (\tilde{\theta}_{n+2})^{i_2} & \cdots & (\tilde{\theta}_{n+2})^{i_n} & (\tilde{\theta}_{n+2})^{-1} \end{vmatrix},$$

where $c_{i_1 \cdots i_{n-1}}(a)$ are some polynomials in a with multi-index $i_1 \cdots i_n$ such that $i_j \leq j$ for all $j = 1, \cdots, n$ and at least one inequality above is strict, i.e. $i_j < j$ for one of j. Thus for each index $i_1 \cdots i_n$, the $n+1$ numbers, $0, i_1, \cdots, i_n$ could only be at most n distinct numbers among $0, 1, \cdots, n$. Thus $0, i_1, \cdots, i_n$

are not distinct numbers from 0 to n. This means that two of $0, i_1, \cdots, i_n$ must be the same. Thus at least two columns of the determinants in the formula (2.7) are the same. Thus the formula (2.7) is zero. Hence

$$V(h, \frac{1}{t-a}) = \frac{1}{v_h} \begin{vmatrix} 1 & \tilde\theta_1 & (\tilde\theta_1)^2 & \cdots & (\tilde\theta_1)^n & (\tilde\theta_1)^{-1} \\ 1 & \tilde\theta_2 & (\tilde\theta_2)^2 & \cdots & (\tilde\theta_2)^n & (\tilde\theta_2)^{-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \tilde\theta_{n+2} & (\tilde\theta_{n+2})^2 & \cdots & (\tilde\theta_{n+2})^n & (\tilde\theta_{n+2})^{-1} \end{vmatrix} \tag{2.8}$$

At last we directly calculate

$$\begin{vmatrix} 1 & \tilde\theta_1 & (\tilde\theta_1)^2 & \cdots & (\tilde\theta_1)^n & (\tilde\theta_1)^{-1} \\ 1 & \tilde\theta_2 & (\tilde\theta_2)^2 & \cdots & (\tilde\theta_2)^n & (\tilde\theta_2)^{-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \tilde\theta_{n+2} & (\tilde\theta_{n+2})^2 & \cdots & (\tilde\theta_{n+2})^n & (\tilde\theta_{n+2})^{-1} \end{vmatrix} \tag{2.9}$$

$$(factor\ out\ each\ (\tilde\theta_i)^{-1}\ from\ i-th\ row) \tag{2.10}$$

$$= \frac{1}{\tilde\theta_1 \cdots \tilde\theta_{n+2}} \begin{vmatrix} \tilde\theta_1 & (\tilde\theta_1)^2 & (\tilde\theta_1)^3 & \cdots & (\tilde\theta_1)^{n+1} & 1 \\ \tilde\theta_2 & (\tilde\theta_2)^2 & (\tilde\theta_2)^3 & \cdots & (\tilde\theta_2)^{n+1} & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \tilde\theta_{n+2} & (\tilde\theta_{n+2})^2 & (\tilde\theta_{n+2})^3 & \cdots & (\tilde\theta_{n+2})^{n+1} & 1 \end{vmatrix} \tag{2.11}$$

$$= (-1)^{n+1} \frac{1}{\tilde\theta_1 \cdots \tilde\theta_{n+2}} \prod_{i>j} (\tilde\theta_i - \tilde\theta_j) \tag{2.12}$$

$$= \frac{1}{h(a)} \prod_{i>j} (\theta_i - \theta_j) \tag{2.13}$$

Combining the calculations in the formulas (2.3), and (2.8)-(2.13), we proved the formula (1.6).

Next we prove the formula (1.7). It is clear that we need to use the technique of "partial fraction" in calculus to reduce the general rational function to that in the form "$\frac{f}{t-a}$". We may assume the $deg(g) = d$ and

$$g = \prod_{i=1}^{d} \eta_i, i = 1, \cdots, d \tag{2.14}$$

where $\eta_i(t) = t - w_i$ is a linear polynomial. By our assumption, w_i are distinct. Let

$$\phi_j(t) = \frac{g}{t - w_j} \tag{2.15}$$

be the monic polynomial of degree $d-1$. Let $\eta_0(t)$ be any linear polynomial that does not vanish at each $w_i, i = 1, \cdots, d$. Then f can be expressed as

$$f(t) = l(t)[\eta_0 \sum_{k=1}^{d} \epsilon_k \phi_k + \epsilon_{d+1} g]. \tag{2.16}$$

for some $\epsilon_k, k=1,\cdots,d$ in $\mathbf{K}$ and some polynomial $l(t)$ of $deg(l(t))\leq n$, where $l(t)$ is any polynomial that divides $f(t)$ and has degree $deg(f)-deg(g)$. Now we use the partial fraction

$$\frac{f}{g}=l(t)[\eta_0\sum_{k=1}^{d}\frac{\epsilon_k}{t-w_k}+\epsilon_{d+1}] \tag{2.17}$$

Then

$$V(h,\frac{f}{g})= V(h,l(t)[\eta_0\sum_{k=1}^{d}\frac{\epsilon_k}{t-w_k}+\epsilon_{d+1}]) \tag{2.18}$$

$$=\sum_{k=1}^{d}\frac{1}{v_h}\begin{vmatrix} 1 & \theta_1 & (\theta_1)^2 & \cdots & (\theta_1)^n & (\frac{l\eta_0\epsilon_k}{t-w_k})|_{\theta_1} \\ 1 & \theta_2 & (\theta_2)^2 & \cdots & (\theta_2)^n & (\frac{l\eta_0\epsilon_k}{t-w_k})|_{\theta_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \theta_{n+2} & (\theta_{n+2})^2 & \cdots & (\theta_{n+2})^n & (\frac{l\eta_0\epsilon_k}{t-w_k})|_{\theta_{n+2}} \end{vmatrix}, \tag{2.19}$$

where the term $V(h,l\epsilon_{d+1})=0$ because the last column of this normalized determinant is a linear combination of the other columns. Now we can apply the formula (1.6) to the normalized determinant in the formula (2.19) to obtain

$$V(h,\frac{f}{g})=-\sum_{k=1}^{d}(\frac{l\eta_0\epsilon_k}{h})|_{w_k} \tag{2.20}$$

Notice

$$f(w_k)=(l\eta_0\epsilon_k\phi_k)|_{w_k}, k=1,\cdots,d \tag{2.21}$$

Using the formula (2.21) to substitute the terms in the formula (2.20), we obtain

$$V(h,\frac{f}{g})=-\sum_{k=1}^{d}(\frac{f}{\phi_k h})|_{w_k}. \tag{2.22}$$

Because

$$g'(w_i)=\phi_i(w_i) \tag{2.23}$$

the above formula (2.22) is exactly the formula (1.7).

(b) At last we use the cofactor expansion of the determinant,

$$V(g,\frac{f}{h}) \tag{2.24}$$

along the last column. That is to expand

$$V(g,\frac{f}{h})=\frac{1}{v_g}\begin{vmatrix} 1 & w_1 & (w_1)^2 & \cdots & (w_1)^n & \frac{f(w_1)}{h(w_1)} \\ 1 & w_2 & (w_2)^2 & \cdots & (w_2)^n & \frac{f(w_2)}{h(w_2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & w_{n+2} & (w_{n+2})^2 & \cdots & (w_{n+2})^n & \frac{f(w_{n+2})}{h(w_{n+2})} \end{vmatrix} \tag{2.25}$$

along the last column. The expansion is exactly

$$\sum_{k=1}^{d}(\frac{f}{g'h})|_{w_k}. \tag{2.26}$$

Combining with the formula (1.7), we proved the anti-symmetry, part (b). ◻

3. Example. In this section, we give some examples that show the applications of the formulas.

Example 3.1 (From the view of Lagrange interpolation). The Lagrange polynomial interpolation is a technique to solve the following question: Given $n+2$ points in a plane $(\theta_i, y_i), i = 1, \cdots, n+2$. Can we find a polynomial function $y = f(x)$ of degree $n+1$ such that

$$y_i = f(\theta_i), i = 1, \cdots, n+2?$$

The classical Lagrange interpolation formula gives an exact formula for $f(x)$ in terms of data (θ_i, y_i), which is called the Lagrange interpolation polynomial ([3]). The reason that formula exists is simply because the number $n+2$ of the free variables is the same as the number $n+2$ of given points (θ_i, y_i).

But if we are asking for a polynomial $y = f(x)$ of degree n (or less) such that the function $y = f(x)$ goes through these points:

$$y_i = f(\theta_i), i = 1, \cdots, n+2, \tag{3.1}$$

then because there are more constraints than the free variables in the system of linear equations (3.1), $f(x)$ does not always exist. Now the question is: what kind of data $(\theta_i, y_i), i = 1, \cdots, n+2$ will always have an interpolation polynomial $y = f(x)$ of degree n that satisfies (3.1)? By the system of linear equations (3.1), the condition on these $(\theta_i, y_i), i = 1, \cdots, n+2$ is exactly that the normalized Vadermonde determinant vanishes:

$$V(h, R) = \frac{1}{v_h}\begin{vmatrix} 1 & \theta_1 & (\theta_1)^2 & \cdots & (\theta_1)^n & y_1 \\ 1 & \theta_2 & (\theta_2)^2 & \cdots & (\theta_2)^n & y_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \theta_{n+2} & (\theta_{n+2})^2 & \cdots & (\theta_{n+2})^n & y_{n+2} \end{vmatrix} = 0, \tag{3.2}$$

provided that θ_i are distinct.

For instance, if y_i are the values of the rational function $\frac{F(t)}{t-a}$ at θ_i with $deg(F) \leq n+1$, then our formula (1.6) says that there is no data $\theta_i, i = 1, \cdot, \theta_{n+2}$ could allow any polynomial $f(x)$ of degree n satisfying the equations (3.1), unless $F(t)$ is divisible by $t-a$, i.e. $F(a) = 0$ if and only if there is a degree n polynomial interpolation for the points $(\theta_i, \frac{F(\theta_i)}{\theta_i - a}), i = 1, \cdots, n+2$.

Example 3.2 The theorem (1.1) comes from our study of certain vector bundles in algebraic geometry. The following example is just a hint of such an application.

In the following, all algebraic varieties are over the algebraic closed field **K**. Let $g = g(t) \in H^0(\mathcal{O}_{\mathbf{P}^1}(r))$, where $\mathbf{P}^1$ is a 1 dimensional projective space over **K**. Assume $g(t)$ has distinct zeros. Also let $h \in H^0(\mathcal{O}_{\mathbf{P}^1}(n+2))$ such that h has distinct zeros and h, g do not have common zeros. Consider the affine variety

$$\Gamma_g \subset H^0(\mathcal{O}_{\mathbf{P}^1}(n+r)) \times H^0(\mathcal{O}_{\mathbf{P}^1}(n)) \times H^0(\mathcal{O}_{\mathbf{P}^1}(n+r)) \tag{3.3}$$

$$\Gamma_g = \{(F, s, f) : F = sg + f\}. \tag{3.4}$$

Let

$$\pi : \Gamma_g \quad \to \quad H^0(\mathcal{O}_{\mathbf{P}^1}(n+r)) \tag{3.5}$$

be the projection to the first component in (3.1). Then the intersection

$$\mathcal{C}_{h,g} = \Gamma_g \cap \{F(\theta_1) = \cdots F(\theta_{n+2}) = 0\} \tag{3.6}$$

is a subvariety of Γ_g. Its projection

$$\pi(\mathcal{C}_{h,g})$$

is a hyperplane in $H^0(\mathcal{O}_{\mathbf{P}^1}(n+r))$. If we choose an affine open set of $\mathbf{P}^1$ such that $h(t), g(t)$ both have non-vanishing leading terms and are monic, then this hyperplane $\pi(\mathcal{C}_{h,g})$ is defined by the normalized "augmented Vandermonde determinant",

$$V(h, \frac{f}{g}) = 0,$$

i.e.

$$\pi(\mathcal{C}_{h,g}) = \{f : V(h, \frac{f}{g}) = 0\} \tag{3.7}$$

In particular, if one of h or g has degree 1 with the only zero $a \in \mathbf{K}$, then by our formulas in the theorem (1.1), $\pi(\mathcal{C}_{h,g})$ is independent of h and g, and

$$\pi(\mathcal{C}_{h,g}) = \{f : f(a) = 0\} \tag{3.8}$$

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