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# ANALYTIC SOLUTIONS FOR THIRD ORDER ORDINARY DIFFERENTIAL EQUATIONS

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**Synopsis**:

This work studies an analytic approach for solving higher order ordinary differential equations (ODEs). We develop alternate techniques for solving third order ODEs and discuss possible generalizations to higher order ODEs. The techniques are effective for solving complex ODEs and could be used in other application of sciences such as physics, engineering, and applied sciences.

# Analytic Solutions for Third Order Ordinary Differential Equations

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**Abstract**

This paper focuses on an analytic approach for solving higher order ordinary differential equations (ODEs). We develop a self-adjoint formulation and integrating-factor techniques to solve third order ODEs. The necessary conditions for ODEs to be self-adjoint are also provided. Under these conditions, we find the analytic solution of the ODEs. The solutions produced in this work are exact unlike numerical solutions which have approximation errors. These techniques may be used as a tool to solve odd order and higher order ODEs.

*Keywords*: Second Order Self-Adjoint ODEs; Third Order Self-Adjoint ODEs; Integrating Factor technique; Ricatti ODE;

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# 1 Introduction

The solution of higher order differential equations (DEs) remains as an intriguing phenomenon for engineers, physicists, mathematicians and researchers. Different modeling techniques have been developed to solve DEs considering its different characteristics. These characteristics indicate the physical dynamism of ordinary differential equations (ODEs) like linear ODEs, non-linear ODEs, partial DEs, stochastic DEs, etc. The equations are used in fluid mechanics, physics, astrophysics, solid state physics, chemistry, various branches of biology, astronomy, hydro-dynamic and hydro-magnetic stability, nuclear physics, applied and engineering sciences. The analytic solutions explain the physical properties and dynamics of the problems in the above-mentioned fields.

In general, solving higher order ODEs are complex since the analytic solutions have to satisfy all the physical conditions governing the equation. Thus numerical methods are mostly used to solve higher order ODE. For instance, Khawaja et al. (2018), used iterative power series of $sech(x)$ to solve non-linear ODEs [1]. Vitoriano R. (2016) also used finite element method to solve PDEs [2]. Mariani and Tweneboah (2016); Mariani et. al. (2017), used Itô's calculus, to solve SDEs [3, 4, 5, 8]. However, the numerical methods also yield several errors while solving higher order ODEs. A characteristic feature of analytic solutions is that they provide exact solution, unlike numerical methods.

In this paper, we develop two analytic techniques for solving third order ODEs namely; self-adjoint formulations and integrating factor techniques. Self-adjoint operators for even order ODEs have been studied in [6]. In this studies, we extend this concept to solve higher order differential equations including odd orders, arguing that this work may serve as a reference for solving other higher order self-adjoint type ODEs. In addition, we discuss the integrating factor type techniques for solving higher order ODEs.

The paper is outlined as follows: In section 2, we will briefly review the background of self-adjoint operators and present some known results. Then we present the techniques for solving third order ODEs in the self-adjoint form. This section also discusses the conditions for a third order ODEs to be in the self-adjoint form. Examples and applications are also presented. In section 3, we discuss the integrating factor theory for solving third order

ODEs. Conclusions are contained in Section 4.

# 2 Self-Adjoint Approach for Solving ODEs

In this section, we present the background for the self-adjoint formulation and techniques for solving ODEs. In Ref. [6], [7], the authors used the self-adjoint formulation to solve even order ODEs. In this study, we will develop a technique for third order ODEs, which can be used as a way to solve third order ODEs. To develop the third order self-adjoint technique, we discuss the necessary conditions for a second order ODE to be in the self-adjoint form.

## 2.1 Second Order ODEs in the Self-Adjoint form

We consider a general second order ODE as follows:

$$u_0(x)y''(x) + u_1(x)y'(x) + u_2(x)y(x) = 0 \tag{2.1}$$

If an operator $\Delta$ is defined as $\Delta = u_0(x)\frac{d^2}{dx^2} + u_1(x)\frac{d}{dx} + u_2(x)$, then (2.1) can be rewritten as

$$\bar{\Delta}y = \frac{d^2}{dx^2}(u_0 y) - \frac{d}{dx}(u_1 y) + u_2 y = 0$$

where $u_0' = u_1$. For a given operator $\Delta$, there exists a corresponding operator $\bar{\Delta}$, known as the adjoint operator associated to $\Delta$. If condition

$$u_0' = u_1 \tag{2.2}$$

is satisfied, we see that $\Delta = \bar{\Delta}$ and we say that the operator $\Delta$ is self-adjoint.

**Definition 2.1.** *A second order ODE* (2.1) *is said to be in the self-adjoint form if and only if:*

$$\Delta y = \bar{\Delta}y = (u_0 y')' + u_2 y = 0$$

*where $u_0(x) > 0$ on $(a, b)$, $u_0'(x)$ and $u_2(x)$ are continuous functions on $[a, b]$ and condition* (2.2) *is satisfied.*

**Example 2.2.** *The Legendre's equation is an example of second order self-adjoint ODE, which is given as*

$$(1-x^2)y'' - 2xy' + l(l+1)y = 0$$

*where $l$ is a constant.*

For more background on the self-adjoint approach, the reader is referred to references in [9, 10, 11, 12].
We now present a theorem that states the conditions for a second order ODE to be in a self-adjoint form [6]. In this case, we present an example of a third order ODE and a way to solve it using second order self-adjoint techniques.

**Theorem 2.3.** *If a second order self-adjoint ODE: $(u_0(x)y')' + u_2(x)y = 0$ satisfies the condition*

$$u_2(x) = \frac{u_0''}{2} - \frac{u_0'^2}{4u_0}, \tag{2.3}$$

*then the solution of the ODE is*

$$y(x) = \frac{1}{\sqrt{u_0(x)}}\,(C_1 x + C_0) \tag{2.4}$$

*where $C_0$ and $C_1$ are constants.*

**Example 2.4.** *Consider the third order ODE as follows:*

$$\left(e^x(y' + \frac{1}{x}y)'\right)' - \frac{e^x}{4}(y' + \frac{1}{x}y) = 0 \tag{2.5}$$

*We assume that $w = y' + \frac{1}{x}y$. So Eq. (2.5) becomes:*

$$\left(e^x w'\right)' - \frac{e^x}{4}w = 0, \tag{2.6}$$

*which is a second order ODE for $g$. We now solve the Eq.* (2.6) *using Theorem 2.3. So we have $u_0(x) = e^x$ and $u_2(x) = -\frac{e^x}{4}$. We observe that* (2.3) *is satisfied as:*

$$u_2(x) = \frac{u_0''}{2} - \frac{u_0'^2}{4u_0} = 0 - \frac{e^{2x}}{4e^x} = -\frac{e^x}{4}$$

*Therefore the solution of (2.6) is $w(x) = \frac{1}{\sqrt{e^x}}(C_1x + C_0)$, then we have:*

$$y' + \frac{1}{x}y = \frac{1}{\sqrt{e^x}}(C_1x + C_0) \tag{2.7}$$

*Now Eq. (2.7) is a linear differential equation. Hence, we solve this equation and obtain the solution of the third order self-adjoint differential equation as follows:*

$$y(x) = C_2xe^{-\frac{x}{2}} + C_3e^{-\frac{x}{2}} + \frac{C_4}{x}e^{-\frac{x}{2}} + \frac{C_5}{x}$$

*where $C_2$, $C_3$, $C_4$, and $C_5$ are constants.*

## 2.2 Third Order ODEs in the Self-Adjoint form

In this subsection, we develop a technique that gives conditions for third order ODEs to be in a self-adjoint form. We present a theorem for solving third order self-adjoint type ODEs, providing an analytic solution. This technique can be used as a generalization to odd order self-adjoint ODEs.
We define a third order linear differential operator as:

$$\Delta = u_0(x)\frac{d^3}{dx^3} + u_1(x)\frac{d^2}{dx^2} + u_2(x)\frac{d}{dx} + u_3(x),\ a < x < b \tag{2.8}$$

where $u_0(x) > 0$ on $(a, b)$, and $u_1(x), u_2(x), u_3(x)$ are continuous functions on $[a, b]$. Applying the operator $\Delta$ on a function $y(x)$, we have: $(u_oy')'' = u_0y''' + 2u_0'y'' + u_0''y'$. Therefore,

$$\Delta y = (u_oy')'' + (u_1 - 2u_0')y'' + (u_2 - u_0'')y' + u_3(x)y(x). \tag{2.9}$$

We now assume the following conditions in order to define the self-adjoint operator,

$$u_1(x) = 2u_0'(x),\ q(x) = u_2(x) - u_0''(x),\ p(x) = u_3(x) - q'(x), \tag{2.10}$$

where $u_1(x)$, $q(x)$, and $p(x)$ are continuous functions on $[a, b]$. So an adjoint operator $\bar{\Delta}$ is defined as corresponding to $\Delta$ in (2.9) as follows:

$$\bar{\Delta}y = (u_oy')'' + q(x)y' + u_3(x)y(x) = 0. \tag{2.11}$$

Using the term $q(x)y' = (qy)' - q'y$, Eq. (2.11) can be rewritten as:

$$\bar{\Delta}y = (u_oy')'' + (q(x)y)' + (u_3(x) - q(x)')y(x) = 0$$

hence

$$\bar{\Delta} y = (u_o y')'' + (q(x)y)' + p(x)y(x) = 0.$$

Therefore, we conclude that $\Delta = \bar{\Delta}$ and the operator $\Delta$ is self-adjoint based on the conditions (2.10).

**Definition 2.5.** *A third order ODE is said to be in the self-adjoint form if and only if:*

$$\Delta y = \bar{\mathcal{L}} y = (u_o y')'' + (q(x)y)' + p(x)y = 0,$$

*where $u_0(x) > 0$ and conditions (2.10) are satisfied.*

**Theorem 2.6.** *If a third order self-adjoint ODE*

$$(u(x)y')'' + (q(x)y)' + p(x)y = 0 \tag{2.12}$$

*verifies the conditions $q = u'' - \frac{2}{3u}(u')^2$ and $p = -\frac{u'''}{3} + \frac{2}{3u}u'u'' - \frac{10}{27u^2}(u')^3$ then the solution to (2.12) is*

$$y(x) = \frac{1}{\sqrt[3]{u^2(x)}} \left( C_1 x^2 + C_2 x + C_3 \right)$$

*where $u(x) > 0$, $p(x)$, $q(x)$ are continuous differentiable functions, and $C_1$, $C_2$, $C_3$ are arbitrary constants.*

*Proof.* We define a third order self-adjoint ODE to be of the form:

$$(u(x)y')'' + (q(x)y)' + p(x)y = 0,$$

where $U(x) > 0$ and continuous. The above equation implies that:

$$y''' + 2\frac{u'}{u}y'' + \frac{u'' + q}{u}y' + \frac{q' + p}{u}y = 0 \tag{2.13}$$

Using the change of variable technique for $y = m(x)n(x)$ in (2.13), where $m(x)$ and $n(x)$ are continuous and differentiable functions, we obtain:

$$\begin{aligned} m'''n &+ \left(3n' + \frac{2u'}{u}n\right) m'' + \left(3n'' + 4\frac{u'}{u}n' + \frac{u'' + q}{u}n\right) m' \\ &+ \left(n''' + 2\frac{u'}{u}n'' + \frac{u'' + q}{u}n' + \frac{q' + p}{u}n\right) m = 0 \end{aligned}$$

Assuming that the coefficient of $m''$, $m'$ , and $m$ are zero, we have:

$$3n' + \frac{2u'}{u}n = 0 \tag{2.14}$$

$$3n'' + 4\frac{u'}{u}n' + \frac{u''+q}{u}n = 0 \tag{2.15}$$

$$\text{and, } n''' + 2\frac{u'}{u}n'' + \frac{u''+q}{u}n' + \frac{q'+p}{u}n = 0 \tag{2.16}$$

Without loss of generality, the corresponding solution to (2.14) is

$$n(x) = e^{-\frac{2}{3}\int \frac{u'}{u}du} = u^{-\frac{2}{3}} = \frac{1}{\sqrt[3]{u^2}}. \tag{2.17}$$

Replacing Eq. (2.17) into (2.15) and (2.16), we obtain:

$$uu'' - \frac{2}{3}(u')^2 - qu = 0 \tag{2.18}$$

$$\frac{2}{3}u''' - \frac{2}{3u}u'u'' + \frac{8}{27u^2}(u')^3 - q' = 0 \tag{2.19}$$

Now, using (2.18) and (2.19), we obtain the corresponding conditions

$$q = u'' - \frac{2}{3u}(u')^2 \tag{2.20}$$

and

$$p = -\frac{u'''}{3} + \frac{2}{3u}u'u'' - \frac{10}{27u^2}(u')^3 \tag{2.21}$$

Finally from Eqs. (2.14), (2.15), (2.16), (2.20) and (2.21), we obtain:

$$nm''' = 0, \text{ then } m''' = 0$$

thus:

$$m(x) = C_1x^2 + C_2x + C_3$$

where $C_1$, $C_2$, and $C_3$ are constants. Hence, the solution of the third order self-adjoint differential equation (2.12) is

$$y(x) = m(x)n(x)$$

and we conclude that:

$$y(x) = \frac{1}{\sqrt[3]{u^2(x)}}\left(C_1x^2 + C_2x + C_3\right).$$

□

Next, we present applications and examples of third order self-adjoint ordinary differential equations using the Definition 2.5 and Theorem 2.6.

**Example 2.7.** *Consider the equation*

$$y'''e^{2x} + \frac{10}{3}e^{2x}y' + \frac{64}{3}e^{2x}y = 0 \tag{2.22}$$

*To obtain the solution, we first simplify equation* (2.23) *as follows:*

$$\left(e^{2x}y'\right)'' + \left(\frac{4}{3}e^{2x}y\right)' + \left(-\frac{8}{27}e^{2x}\right)y = 0 \tag{2.23}$$

*We assume $u(x) = e^{2x}$. Now we check the conditions of Theorem 2.6 to solve the ODE* (2.23)*:*

$$q = u'' - \frac{2}{3u}(u')^2 = \frac{4e^{2x}}{3}$$

$$\text{and, } p = -\frac{u'''}{3} + \frac{2}{3u}u'u'' - \frac{10}{27u^2}(u')^3 = -\frac{8e^{2x}}{27}$$

*Since the conditions of Theorem 2.6 are satisfied, we can obtain the analytic solution of our ODE as*

$$y(x) = \frac{1}{\sqrt[3]{e^{4x}}}\left(C_1x^2 + C_2x + C_3\right)$$

**Example 2.8.** *Consider the equation*

$$\left(x^4y'\right)'' + \left(\frac{4x^2}{3}y\right)' + \left(\frac{8x}{27}\right)y = 0 \tag{2.24}$$

*In this example, $u(x) = x^4$. Now we check the conditions of Theorem 2.6 to solve the ODE* (2.24)*:*

$$q = u'' - \frac{2}{3u}(u')^2 = \frac{4x^2}{3}$$

$$\text{and } p = -\frac{u'''}{3} + \frac{2}{3u}u'u'' - \frac{10}{27u^2}(u')^3 = \frac{8x}{27}$$

*Since the conditions are satisfied, from Theorem 2.6 the analytic solution is:*

$$y(x) = \frac{1}{\sqrt[3]{x^8}}\left(C_1x^2 + C_2x + C_3\right)$$

# 3 Integrating Factor Approach for Solving ODEs

In this section, we present the integrating factor technique for solving second and third order ODEs. The conditions of second order ODEs help us to develop the integrating factor technique for third order ODEs. We present some examples of third order ODEs that have been solved using this technique. The results obtained can be generalized for integrating factor technique for third order ODEs.
We now begin our discussion with the Integrating Factor technique for the second order ODEs.

## 3.1 Integrating factor technique for second order ODEs

Given a second order ODE, we can obtain the associated Ricatti ODE for a variable $b$. The Ricatti equation is obtained by using the coefficients of the second order ODE. The particular solution of $b$ helps us to obtain an integrating factor of the second order ODE. Using the Integrating Factor we can obtain a particular solution of the second order differential equation.

We consider a general second order ODE,

$$y'' + U(x)y' + V(x)y = W(x) \tag{3.1}$$

A second order ODE can be expressed as:

$$(r(y' + by))' = rW(x), \tag{3.2}$$

where $r$ is an integrating factor. This equation can be rewritten as:

$$y'(r' + rb) + y(r'b + rb') + ry'' = rW(x). \tag{3.3}$$

After multiplying (3.1) by $r$ and comparing the coefficients from (3.1) and (3.3), we have:

$$r' + rb = rU(x) \tag{3.4}$$

$$r'b + rb' = rV(x) \tag{3.5}$$

Using the above two equations, we obtain the Ricatti equation for $b$:

$$b' = V(x) - bU(x) + b^2. \tag{3.6}$$

If we know the homogeneous solution $y_1$ of (3.1), then $-\frac{y_1'}{y_1} = b$ is a solution of the Ricatti equation $b' = b^2 + V - bU$ and if $r$ in (3.4) is given by $\frac{r'}{r} = U - b$, we obtain that $r$ is an integrating factor of (3.1) and this allows us to obtain a particular solution.

**Example 3.1.** *Consider a second order ODE for any constant $k$,*

$$y'' - xy' + y = kx^2 \tag{3.7}$$

*Comparing this equation with* (3.1), *we observe that $U = -x$, $V = 1$, and $W = kx^2$.*

*The Ricatti equation for $b$ is given by:*

$$b' = V - bU + b^2. \tag{3.8}$$

*Knowing that $y_1 = x$ is a solution of the homogeneous equation associated to* (3.7) *i.e. $y'' - xy' + y = 0$, as $b = -\frac{y_1'}{y_1} = -\frac{1}{x}$ then $b = -\frac{1}{x}$ is a particular solution of (3.8). From* (3.4) *and* (3.5) *we obtain:*

$$\frac{r'}{r} = U - b$$

*then*

$$\frac{r'}{r} = -x + \frac{1}{x} \tag{3.9}$$

*Integrating both sides of* (3.9), *we have*

$$\ln r = -\frac{x^2}{2} + lnx + C.$$

*Without loss of generality, $r = e^{-x^2/2}x$.*

*Now plugging $r, b$ and $W$ into* (3.2) *and integrating both sides we obtain*

$$xe^{-x^2/2}\left(y' - \frac{1}{x}y\right) = \int kx^3e^{-x^2/2}dx = -ke^{-\frac{x^2}{2}}(x^2+2) + C.$$

*This implies that*

$$y' - \frac{1}{x}y = -\, k(x + \frac{2}{x}\Big) + \frac{Ce^{x^2/2}}{x}.$$

*Now we assume $C = 0$, as we seek to find the particular solution. Using first degree order integrating factor theory, we obtain the solution as:*

$$y = k_1 - kx^2$$

*where $k_1$ and $k$ are constants.*

## 3.2 Integrating factor technique for third order ODEs

In this section, we present the integrating factor techniques for solving third order ODEs, providing the following theorem.

**Theorem 3.2.** *Given*

$$y''' + U(x)y'' + V(x)y' + W(x)y = f(x), \tag{3.10}$$

*if we know a solution to the associated integrating factor equation*

$$r''' - Ur'' + (V - 2U')r' + (V' - U'' - W)r = 0 \tag{3.11}$$

*or alternatively, a solution to*

$$y''' + 2Uy'' + (U' + U^2 + V)y' + (V' - W + VU)y = 0, \tag{3.12}$$

*then we can find a particular solution to* (3.10)*.*

*Proof.* We solve (3.10) by formulating

$$[r(y'' + by' + ay)]' = rf(x) \tag{3.13}$$

where $r$ satisfies (3.11). Multiplying (3.10) by $r$ we obtain

$$r(y''' + U(x)y'' + V(x)y' + W(x)y) = rf(x). \tag{3.14}$$

From (3.13) and (3.14), we get

$$r'(y'' + by' + ay) + r(y'' + by' + ay)' = rf(x). \tag{3.15}$$

Expanding we get

$$y'''r + y''(r' + rb) + y'(r'b + rb' + ra) + y(r'a + ra') = rf(x). \tag{3.16}$$

From (3.13)-(3.16), and using the fact that Eq. (3.14) = Eq. (3.16) we obtain the following three conditions:

$$r' + rb = rU \tag{3.17}$$

$$r'b + rb' + ra = rV \tag{3.18}$$

$$r'a + ra' = rW \tag{3.19}$$

From (3.17) we have that:

$$\frac{r'}{r} = U - b \tag{3.20}$$

and from (3.18):

$$\frac{r'}{r} = \frac{V - b' - a}{b}. \tag{3.21}$$

So from (3.20) and (3.21), $U - b = \frac{V-b'-a}{b}$ then we have that $(U - b)b = V - b' - a$ and solving for $a$, we obtain

$$a = V - b' - b(U - b). \tag{3.22}$$

Finally from (3.19), we have $r'a = r(W - a')$ and so

$$\frac{r'}{r} = \frac{W - a'}{a} \tag{3.23}$$

Combining with (3.20), we obtain

$$\frac{r'}{r} = \frac{W - a'}{a} = U - b. \tag{3.24}$$

Thus $a(U - b) = W - a'$. Replacing in (3.24) $a$ and $a'$, we get

$$\left[V - b' - b(U - b)\right](U - b) = W - \left[V - b' - b(U - b)\right]'. \tag{3.25}$$

From (3.25), we obtain

$$b'' = V' - W + VU + b(-V - U' - U^2) + 3bb' - 2b'U + 2Ub^2 - b^3). \tag{3.26}$$

Using the change of variable $b = -\frac{y'}{y}$, we have that

$$b' = -\frac{y''}{y} + \frac{y'^2}{y^2} \tag{3.27}$$

Then replacing $b = -\frac{y'}{y}$ into (3.27), we conclude that $b' = -\frac{y''}{y} + b^2$, and

$$b'' = -\frac{y'''}{y} + \frac{y'}{y}\frac{y''}{y} + 2bb'. \tag{3.28}$$

Since $b' = -\frac{y''}{y} + b^2$, replacing $\frac{y'}{y} = -b$ into (3.28) we have

$$b'' = -\frac{y'''}{y} - b\frac{y''}{y} + 2bb'$$

and using $b' = -\frac{y''}{y} + b^2$ we get $\frac{y''}{y} = b^2 - b'$, therefore

$$b'' = -\frac{y'''}{y} - b^3 + 3bb'. \tag{3.29}$$

Considering the homogeneous differential equation associated to (3.10) i.e.

$$y''' + U(x)y'' + V(x)y' + W(x)y = 0,$$

we obtain

$$\frac{y'''}{y} = -U(x)\frac{y''}{y} - V(x)\frac{y'}{y} - W(x). \tag{3.30}$$

Replacing $\frac{y''}{y} = b^2 - b'$ and $\frac{y'}{y} = -b$ in (3.30):

$$\frac{y'''}{y} = -U(x)\left[b^2 - b'\right] + V(x)b - W(x)$$

and replacing the last expression into (3.29) we get

$$b'' = W(x) + b^2U(x) + 3bb' - b^3 - bV(x) - b'U(x). \tag{3.31}$$

Therefore (3.31) represents the Ricatti equation associated to (3.10). On the other hand, if we replace (3.29) into (3.26) and use the relations $b = -\frac{y'}{y}$ and $b' = -\frac{y''}{y} + b^2$ we get:

$$y''' + 2Uy'' + (U' + U^2 + V)y' + (V' - W + VU)y = 0 \tag{3.32}$$

which is the corresponding homogeneous equation associated to (3.26).

We recall from Eqs. (3.17) and (3.18) that $rb = rU - r'$ and $(rb)' + ra = rV$ respectively. So from (3.17), $(rb)' = r'U + rU' - r''$ and replacing this into (3.18), we have:

$$ra = rV - r'U - rU' + r''. \tag{3.33}$$

From (3.19), $(ra)' = rW$, so taking the derivative of (3.33), we obtain the integrating factor equation

$$r''' - r''U + r'(V - 2U') + r(V' - U'' - W) = 0 \tag{3.34}$$

Thus if we have a solution for the integrating factor equation (3.34), we can find a particular solution for

$$y''' + U(x)y'' + V(x)y' + W(x)y = f(x)$$

and conversely, if we know a particular solution for (3.10), we can obtain a solution for the integrating factor equation (3.34). In fact, knowing one solution for

$$y''' + U(x)y'' + V(x)y' + W(x)y = 0,$$

we can find a solution for

$$r''' - Ur'' + r'(V - 2U') + r(V' - U'' - W) = 0$$

Hence the proof. $\square$

**Remark 3.3.** *If we know one solution of* (3.32)*, then by using* $b = -\frac{y'}{y}$*, we know the solution of* (3.26) *and conversely, if we know one solution of* (3.26)*, we know the solution of* (3.32)*. In addition, if we know the solution of* (3.31)*, we know one solution for the homogeneous ODE associated to* (3.10)*.*

**Example 3.4.** *Consider the third order ODE*

$$r''' + x^2r'' + 6xr' + 6r = 0, \tag{3.35}$$

*In our analysis, we use two solutions of the above equation, that is,* $r_1 = x^2e^{-x^3/3}$ *and* $r_2 = x^2e^{-x^3/3}\int \frac{e^{x^3/3}}{x^3}dx$*.*
*Comparing Eq.* (3.35) *to Eq.* (3.34)*, we see that*

$$-U = x^2, V - 2U' = 6x, \text{ and } V' - U'' - W = 6.$$

*Solving for* $U, U', U'', V, V'$ *and* $W$*, the corresponding equation for* (3.10) *is*

$$y''' - x^2y'' + 2xy' - 2y = f(x).$$

*Since* $\frac{r'}{r} = U - b$ *from* (3.17)*, we can solve for* $b$ *using one particular solution* $r = e^{-x^3/3}x^2$*. Now* (3.17) *gives us:* $b = -\frac{2}{x}$*, and Eq.* (3.22) *gives:* $a = \frac{2}{x^2}$*. So replacing* $r, b$ *and* $a$ *in* (3.13) *and integrating both sides we get:*

$$e^{-x^3/3}x^2\left(y'' - \frac{2}{x}y' + \frac{2}{x^2}y\right) = \int e^{-x^3/3}x^2f(x)dx.$$

*We assume that* $f(x) = c$, *then*

$$e^{-x^3/3}x^2\left(y'' - \frac{2}{x}y' + \frac{2}{x^2}y\right) = -ce^{-x^3/3} + k$$

*which becomes,*

$$y'' - \frac{2}{x}y' + \frac{2}{x^2}y = -\frac{c}{x^2} + k\frac{e^{x^3/3}}{x^2}.$$

*In order to obtain a solution of the equation above, we use Euler method for the homogeneous equation as:* $y'' - \frac{2}{x}y' + \frac{2}{x^2}y = 0$ *or*

$$x^2y'' - 2xy' + 2y = 0. \tag{3.36}$$

*Setting* $y = x^r$ *we get the fundamental solutions to be* $y_1 = x^2$ *and* $y_2 = x$. *Hence, the general solution for the homogeneous part is* $y_h = c_1x^2 + c_2x$. *To find a particular solution we implement again the technique used in the previous example.*
*Now we consider*

$$y'' - \frac{2}{x}y' + \frac{2}{x^2}y = g(x) \tag{3.37}$$

*with* $g(x) = -\frac{c}{x^2} + k\frac{e^{x^3/3}}{x^2}$. *Using that* $y = x$ *and* $b = -\frac{y'}{y}$ *in Eq.* (3.2) *with* $g(x)$ *instead of* $W(x)$, *we obtain:*

$$(r(y' + by))' = rg(x) \tag{3.38}$$

*From* (3.37), *we have* $U = -\frac{2}{x}$ *and* $b = -\frac{y'}{y} = -\frac{1}{x}$. *So* $\frac{r'}{r} = -\frac{1}{x}$, *hence* $r = x^{-1}$. *Thus replacing* $r$ *and* $b$ *into* (3.38), *we have:*

$$\left(x^{-1}\left(y' + (-\frac{1}{x})y\right)\right)' = \frac{1}{x}g(x), \text{ then } \frac{1}{x}\left(y' - \frac{1}{x}y\right) = \int \frac{1}{x}g(x)dx + \tilde{k}$$

*which is solvable by using the first degree order integrating factor technique.*

**Example 3.5.** *Consider the third order ODE*

$$xy''' - (x^2 + 2)y'' + 2xy' - 2y = 0, \tag{3.39}$$

*This equation can be written as:*

$$xy''' - x^2y'' - 2y'' + 2xy' - 2y = 0 \Rightarrow x(y''' - xy'') - 2(y'' - xy' + y) = 0 \tag{3.40}$$

*To obtain a solution of this equation, we assume that* $v = y'' - xy' + y$ *and* $v' = y''' - xy''$. *So Eq. 3.40 gives us the following form:*

$$xv' - 2v = 0$$

*which is a first order ODE for* $v$. *We obtain a solution for* $v$ *as follows:* $\frac{v'}{v} = \frac{2}{x}$. *Then integrating both sides we obtain:* $lnv = 2lnx + lnk \Rightarrow v = kx^2$. *Replacing* $v$, *we obtain:*

$$y'' - xy' + y = kx^2$$

*This equation has been solved in Example 3.1. So we can easily obtain the solution of the Eq.(3.39).*

# 4 Conclusions

In this work, we developed self-adjoint formulations and integrating-factor techniques for solving third order ODEs. We studied these techniques for the second order ODEs, and generalized the techniques for third order ODEs. The solutions obtained in this study are exact, therefore the techniques can be used as alternative to numerical methods. In addition, these techniques will serve as reference for further studies in finding solutions to higher order ODEs.

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