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THE ROLE OF DEFINITIONS IN GEOMETRY FOR PROSPECTIVE MIDDLE AND HIGH SCHOOL TEACHERS

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ABSTRACT. Does verbalizing a key definition before solving a related problem help students solving the problem? Perhaps surprisingly, the answers turns out to be NO, based on this yearlong experiment conducted with mathematics education majors and minors in two geometry classes. This article describes the experiment, the results, suggests possible explanations, and derives some peda-gogical conclusions that are useful for teachers of any mathematics or science course.

Keywords: concept definition, concept image, proof writing, problem solving, secondary mathematics teachers.

1. INTRODUCTION

Definitions play a very important role in mathematics, the learning of mathematics, and the teaching of mathematics alike. However, knowing a definition is not equivalent with knowing a concept. To become effective problem solvers, students need to develop an accurate concept image. The formal definition is only a small part of the concept image, which also includes examples and non-examples, properties and connections to other concepts. The distinction between concept definition and concept image has been examined by many research studies (e.g., Edwards, 1997; Vinner and Dreyfus, 1989; Tall, 1992; Vinner, 1991).

Another group of articles analyze, argue for, and provide examples of engaging students in the construction of definitions (e.g., Zandieh and Rasmussen, 2010; Herbst et al., 2005; Johnson et al., 2014; Zaslavski and Shir, 2005). These opportunities to define concepts and to analyze, probe, and compare definitions, are especially important to preservice teachers of mathematics. They will be making the crucial pedagogical decisions every day in their classrooms what words to use in defining a concept, how many examples and non-examples to give, and how to assess their students' understanding. There seems to be evidence that many mathematics teachers' understanding of the role and importance of definitions is inadequate in this area (Johnson et al., 2014 and Edwards and Ward, 2004). As Poincare` (1914) pointed out in a book written over 100 years ago:

> What is a good definition? For the philosopher or the scientist, it is a definition which applies to all the objects to be defined, and applies only to them; it is that which satisfies the rules of logic. But in education it is not that; it is the one that can be understood by the pupils. (p. 117)

In this paper I describe a series of experiments conducted in two college level geometry classes for prospective mathematics teachers. In both classes, I regularly engage my students in creating definitions of familiar concepts throughout the semester. We share them, compare equivalent definitions, and emphasize the benefits of a definition being minimal, accurate, and elegant. Naturally, I always have a question or two on every exam where students have to write a definition of a given concept. These concepts and their definitions have been learned in class, and students are encouraged not to memorize the textbook's definitions, but rather to use their own words and carefully construct a verbal definition satisfying the requirements. This is a small part of the exam (about

FIGURE 1. One of the experimental problems

Exam for control group	Exam for experimental group
Problem 3: Prove that if the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram. *(10 points)*	Problem 3: a. Give a definition of a parallelogram. *(5 points)* b. Prove that if the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram. *(10 points)*

10%), the majority of the exams is still proofs, constructions and calculations. The main question I was trying to answer with the experiment was: Does the task of writing out the definition of a key concept before a problem help students solve the problem? The secondary question was: Do students who score higher on the definitions perform better on the related problem?

2. METHODOLOGY

I have collected data from fourteen experiments conducted over five exam sessions in geometry classes each with 18–22 students who agreed to participate in the study. In each experiment a randomly chosen half of the class were given the extra task of writing out a definition of one key concept before solving the related problem (usually writing a proof). The other half of the class served as the control group, they only had to solve the same problem and did not get prompted to write out the definition. To keep the experiment fair to all students (and for the point totals to add up to 100 for everybody) the roles have been reversed in another problem of the same exam. The pairing of the experiments guaranteed that everybody was in the treatment group once and in the control group once (or twice each, in case of the final exam). Figure 1 shows an example of one experiment: the two versions of a problem given to the two groups.

The definitions were graded on a 5 point scale based on the following rubric. The examples are collected from the task to define the "midsegment of a triangle", which was assigned to both classes, so I had over 20 definitions written for it.

5 points: correct, and minimal or barely-not-minimal definition
Ex: "*The line segment that connects two midpoints of the sides of the triangle.*"

4 points: correct, but not minimal or slightly incomplete definition
Ex: "*The line segment connecting two midpoints of a triangle, which is parallel to the third side.*"

3 points: student probably has the right concept image, but the wording of the definition is lacking major parts
Ex: "*A line between two midpoints.*"

2 points: student has probably an incorrect concept image, for example the definition refers to a special case only
Ex: "*The horizontal line segment in the middle of the triangle.*"

1 point: hardly any part of the definition is correct (maybe a correct picture is drawn but the words are incorrect)
Ex: "*The line segment that connects one midpoint to the vertex.*"

0 points: no relevant work is found (usually the question is left blank or the concept totally misunderstood)

Ex: *“Bisects the angles.”*

While the problems following the definitions had different point values on the different exams, for the sake of the statistical analysis each was converted to a percentage. I graded the proofs rather leniently, mostly looking for the logical deductions and sequence of steps, without any formal requirement on style, grammar, complete sentences, etc. Similarly, if the experimental problem was a compass and straightedge construction, accuracy, efficiency or neatness was not required, just visible markings that demonstrate the theoretical correctness of the construction.

The data were analyzed at the level of experimental problem, with the primary predictor of having had a question with the matching definition as determined by the randomization group. Descriptive statistics were computed by averaging the problem score separately for both groups and each problem. The primary analysis used mixed effects linear regression with a random subject effect to adjust for the study design, in which 2-4 experimental problems were present on each exam as described above. In addition to the primary predictor of having been asked for the matching definition, the fixed effects in the model included indicators for each problem to adjust for differences in difficulty, and the score achieved on the other exam problems (as a percentage of maximum attainable) to adjust for student ability. In a secondary analysis using data from responses where the matching definition was requested, the effect of the definition score on the problem score was evaluated using a similar mixed effects model with a random subject effect and fixed effects for the definition score and problem indicator. This analysis was conducted separately among students who scored above or below the median (65%) on other problems.

Estimates are reported as value ± standard error. A two-sided 5% significance level is used. All analyses were performed in R 3.1.2 (R Foundation for Statistical Computing, Vienna, Austria).

FIGURE 2. Mean gain in score associated with being asked for a matching definition.

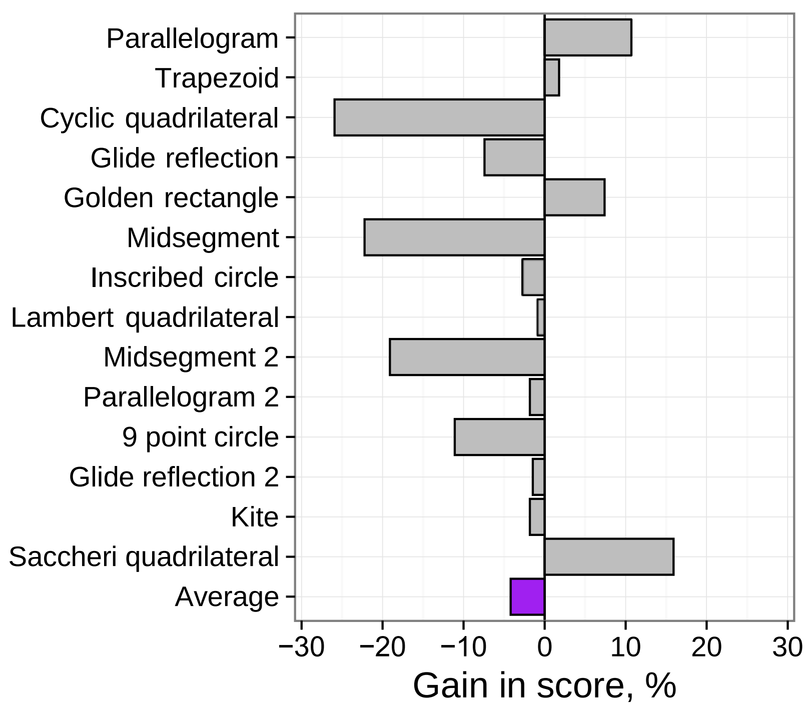
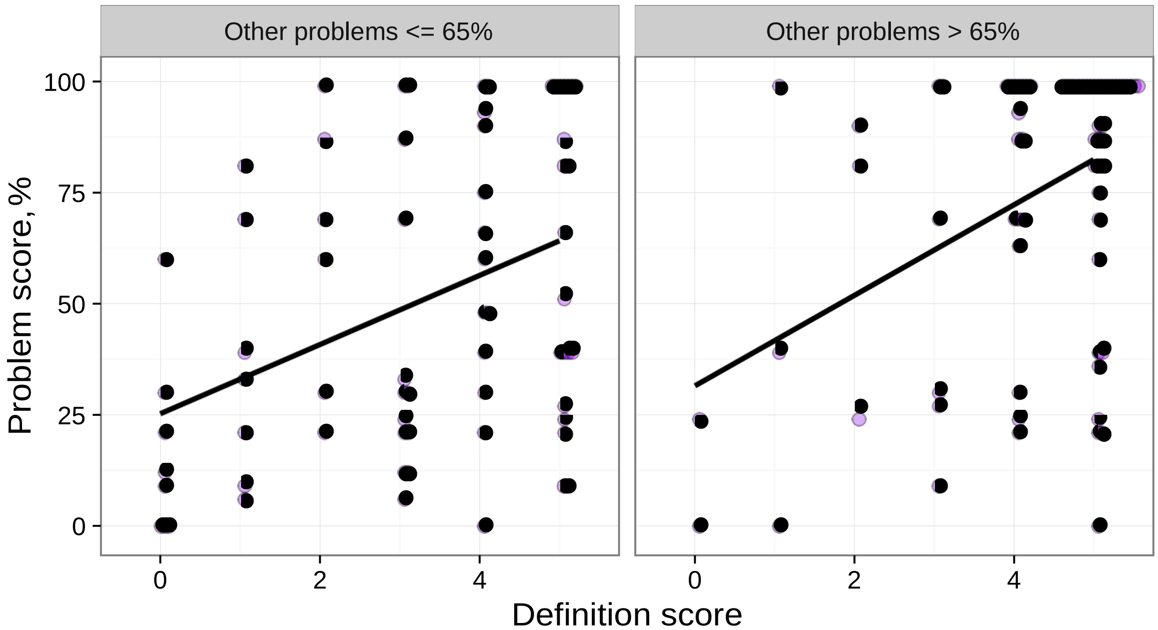


FIGURE 3. Relationship between knowledge of the definition and score obtained on the problem associated with the definition.



3. RESULTS

Data from a total of 274 solved problems from 97 completed exams was collected, with half of them having a matching definition. Some individuals participated in the study more than once, however due to the anonymous data collection these could not be identified.

On the main question, whether the task of writing out a formal definition helps the problem solving, not only did I not find that the group that was asked to write out definitions scored higher on the related problems, they actually scored slightly lower than the control group on average. As Figure 2 shows, out of the 14 separate experiments, the group that had to write definitions scored higher in only 4 cases, the other 10 times the control group scored higher. The overall effect of being asked to write definitions was negative ($-6.6 \pm 4.3\%$, $p = 0.13$), although the difference is not statistically significant. This result was somewhat of a surprise, but supports other research showing the disconnect between concept definition and concept image in students mind (Edwards, 1997; Vinner and Dreyfus, 1989; Tall, 1992; Vinner, 1991).

On the second question, whether students who know the definitions better score higher, the results showed a strong positive correlation. This was true both for weaker students (defined as students whose overall exam score was $\leq 65\%$), and stronger students (with scores over 65%), as seen in the scatterplots on Figure 3. The regression lines show that each extra point on the definition question is associated with almost 10% higher score on the related problem. This correlation remained true when controlled for problem difficulty, with an estimated increase of $9.9 \pm 2.8\%$ per point among weaker and $7.7 \pm 3.3\%$ per point among stronger students ($p < 0.001$ and $p = 0.021$, respectively). This result was expected, students who know the definition have a better chance of completing the proof, while those who cannot form a correct verbal definition of a concept have little chance of solving the related problem. On the other hand, this result also supports the importance of defining concepts as a valuable skill.

4. CONCLUSIONS

4.1. **The role of definitions.** The results of the experiment support the well accepted fact that the formal definition of a concept plays a role, but not a huge role in building the correct concept image. While some of us mathematicians may believe that starting a proof with the definition of the related concept is natural, for prospective secondary teachers this does not seem to be natural or even helpful. Their proof writing and problem solving ability is usually not sophisticated enough to gain any advantage by writing out concept definitions.

4.2. **Implications for Teaching.** College professors must be aware that giving a definition to students does not mean that they have learned the concept. Even if the students can memorize and verbally repeat the definition, they are a long way from being able to use that concept in problem solving situations. Later, when the concept image gets clear, students might have a hard time to isolate the definition from properties and theorems learned about the concept. A few minutes of class time should be devoted to explaining what a mathematical definition is or is not.

Teachers' understanding of what a mathematical definition is and how it is used affects their curricular and pedagogical decisions (Borasi, 1992, and Zazkis and Leikin, 2008). Future teachers must learn to state definitions correctly and to connect it to accurate concept images. Moreover, they must learn to use definitions in problem solving and proof writing. Their college level preparation therefore must include many opportunities to practice this, and mathematicians should model and emphasize appropriate use of definitions.

For example, many students write "by definition" at the beginning or the end of a sentence in proofs. This habit may originate from mimicking a professor's proof, but while the professor had a specific definition in mind, students just use the phrase without thinking. I teach them to use the phrase "by definition" only when they would able to answer the question "by definition of what?" without much thinking.

4.3. **Further research.** It would be interesting to repeat a similar experiment in other areas of mathematics, or other scientific fields. Further insight could be obtained by conducting in-depth interviews about students understanding of the role of definition in geometry. Finally, it would interesting to see how teachers understanding of the role of definitions evolves after they graduate and spend 3-5 years teaching mathematics.

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