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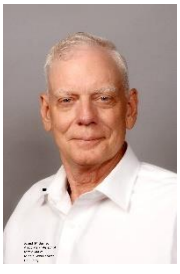
THE TELESCOPING SERIES-EXAMINED



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The Telescoping Series-Examined

Synopsis:

The telescoping Series is a method for examining the convervence of infinite series of the form: This method, combined with partial fraction decomposition, is frequently effective. This paper will examine when the telescoping series is appropriate and simplify the summation process involving rational and transcendental functions. The telescoping series is a method for examining the convervence of infinite series of the form: this method, combined with partial fraction decomposition, is frequently effective.

The Telescoping Series-Extended

The telescoping series is a method for examining the convervence of infinite series of the form:

$\sum_1^{\infty}\left[f(x)-g(x)\right]$

This method, combined with partial fraction decomposition, is frequently effective. This paper will examine when the telescoping series is appropriate and simpify the summation process involving rational and transcendental functions.

Example of Telescoping Series:

$$\sum_1^{\infty} \frac{1}{n(n+1)}$$

$$\sum_1^{\infty} \frac{1}{n(n+1)} = \frac{A}{n} - \frac{B}{n+1} =$$

$$\sum_1^{\infty} \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} =$$

$$f(n) = \frac{1}{n} \text{ and } f(n+1) = \frac{1}{n+1}$$

$\therefore$ the telescoping series applies and

$$\sum_1^{\infty} \frac{1}{n(n+1)} = f(1) + \lim_{n\to\infty} \frac{1}{n+1} = f(1) = 1$$

This method is applicable when
$g(x) = f(x+k)$
where k is a positive integer.

Students frequently find this method hard to understand because they cannot determine whether it is applicable without a time consuming listing of the first few terms of the sequence until a pattern is established and an induction decision can be made.

Realizing that $g(x) = f(x+k)$ can save time and indicate that the series is telescoping.

Suppose that $g(x) = f(x+k)$.

$$\sum_1^{\infty}\left[f(x) - g(x)\right] =$$
$$f(1) - f(1+k) +$$
$$f(2) - f(2+k) +$$
$$f(3) - f(3+k) + \ldots$$
$$f(k) - f(k+k) +$$
$$f(k+1) - f(2k+1) +$$
$$f(k+2) - f(2k+2) +$$
$$f(k+3) - f(2k+3) +$$
$$f(k+4) - f(2k+4) +$$
$$f(k+5) - f(2k+5) + \ldots +$$
$$f(2k) - f(3k) +$$
$$f(2k+1) - f(3k+1) +$$
$$f(2k+2) - f(3k+2) + \ldots$$

Suppose that $g(x) = f(x+k)$.

$$\sum_1^\infty \left[f(x) - g(x) \right] =$$

$$f(1) - f(1+k) +$$
$$f(2) - f(2+k) +$$
$$f(3) - f(3+k) + \ldots$$
$$f(k) - f(k+k) +$$
$$f(k+1) - f(2k+1) +$$
$$f(k+2) - f(2k+2) +$$
$$f(k+3) - f(2k+3) +$$
$$f(k+4) - f(2k+4) +$$
$$f(k+5) - f(2k+5) + \ldots +$$
$$f(2k) - f(3k) +$$
$$f(2k+1) - f(3k+1) +$$
$$f(2k+2) - f(3k+2) + \ldots$$
$$f(n-k-3) - f(n-3)$$
$$f(n-k-2) - f(n-2)$$
$$f(n-k-1) - f(n-1)$$
$$f(n-k) - f(n) +$$
$$f(n-k+1) - f(n+1) +$$
$$f(n-k+2) - f(n+2) +$$
$$f(n-k+3) - f(n+3) + \ldots +$$
$$f(n-3) - f(n-3+k)$$
$$f(n-2) - f(n-2+k) +$$
$$f(n-1) - f(n-1+k) +$$
$$f(n) - f(n+k)$$

$$\left[\sum_1^{\infty} f(n) - f(n+k)\right] =$$
$$f(1) + f(2) + f(3) + \cdots f(k)$$
$$- f(n+1) - f(n+2) - f(n+3) - \cdots f(n+k) =$$
$$\sum_{i=1}^{k} f(i) - \sum_{i=1}^{k} f(n+i)$$

$\therefore$ The convergence of the infinite series can be determined by finding

$$\lim_{x\to\infty} \sum_{n=1}^{k} f(n) - \lim_{n\to\infty} \sum_1^{k} f(n+k)$$

If this limit is 0 then the series converges to $\sum_{i=1}^{k} f(i)$.

If $\lim_{n\to\infty} \sum_1^{k} f(n+k) \in \Re$,

then the series converges to $\sum_{i=1}^{k} f(i) - \sum_{i=1}^{k} f(n+i)$.

If the series is telescoping then the sum of the series can be found by finding the sum of a finite number of terms.

Example Two:
Telescoping series with k=1

$$\sum_1^{\infty} \frac{20n-20}{4n^2-8nn+3} = \left[\sum_{n=1}^{\infty} \frac{5}{2n-3} + \frac{5}{2n-1}\right]$$

$$\sum_1^{1} \frac{5}{2n-3} + \lim_{n\to\infty} \frac{5}{2n-1} = -5$$

Example Three:
Telescoping series with k=2

$$\sum_{1}^{\infty} \frac{12}{9n^2+30n+16} = \left[\sum_{n=1}^{\infty} \frac{2}{3n+2} - \frac{2}{3n+8}\right]$$

$$\sum_{1}^{2} \frac{5}{3n+2} - \lim_{n\to\infty} \frac{2}{3n+8} - \lim_{n\to\infty} \frac{2}{3n+11} = \frac{13}{8}$$

Example Four:
Telescoping series transcendental function, k=1:

$$\sum_{1}^{\infty}\left[\frac{1}{\ln en} - \frac{1}{\ln(en+e)}\right] =$$

$$\sum_{n=1}^{1} \frac{1}{\ln en} = 1$$

Example Five:
Telescoping series transcendental function, k=2:

$$\sum_{n=0}^{\infty}\left[\arctan n - \arctan(n+2)\right] =$$

$$\sum_{n=0}^{1} \arctan n - \lim_{n\to\infty} \arctan n \lim_{n\to\infty} \arctan(n+1) =$$

$$\frac{\pi}{4} - \pi = \frac{3}{4}\pi$$

Summary:

The Telescoping series is an effective technique for finding the sum of an infinite series and can be used when the series is of the form $\left[\sum_1^{\infty} f(n) - f(n+k)\right]$ where k is a natural number.
It can be used on both rational algebraic and transcendental functions.

The Telescoping Series-Extended

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Middle Georgia State University

2018 - STEM / STEAM Education Conference

Date of Conference:

June 6, 7, & 8, 2018

Prince Waikiki

Definition of a sequence

- A sequence is a function defined on the natural numbers.

$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$ harmonic

$\left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\right\}$ geometric

$\{1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots\}$ Fibonacci

Infinite Series

$\left\{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...\right\}$ divergent harmonic series

$\left\{1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+...\right\}$ convergent geometric series

$\sum_{i=1}^{\infty} a_i$ where the a_i's are the terms of a sequence in an infinite series.

Convergence of an Infinite Series

- An infinite series is said to converge if there is some real number that the sequence of partial sums gets closer and closer to as n goes to infinity.
- Infinite series are used in term by term integration, in finding sine, natural log, etc. on a calculator.

It is important to be able to determine whether an infinite series converges.

There are many ways to try to determine convergence. Some of the more relevant are the geometric series, the basic comparison test, the ratio test and others.

This paper will analyze the telescoping series method.

The Telescoping Series

The telescoping series is an infinite series of the form $\sum_{n=1}^{\infty}\left[f(n)-g(n)\right]$ where all but a finite number of the terms cancel out.
This method, sometimes combined with partial fraction decomposition, is frequently effective in determining not only convergence, but also the real number to which the series converges.

This paper will examine when the telescoping series is appropriate, simplify the summation process (including rational and transcendental functions) and extend the application.

Students frequently find the telescoping series concept difficult to understand because they cannot determine whether it is applicable without a time consuming listing of the first few terms of the sequence until a pattern is established and an inductive decision can be made.

Realizing that $g(n) = f(n+k)$ can save time and indicate that the series is telescoping.

Example One:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

$$\frac{1}{n(n+1)} \leq \frac{1}{n^2} \forall n > 0$$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series.

$\therefore \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges by the

Direct Comparison Test.

Example One:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

by Partial Fraction decomposition;

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum\left(\frac{\mathrm{A}}{n} - \frac{\mathrm{B}}{n+1}\right) =$$

$$\frac{1}{n(n+1)} = \frac{\mathrm{A}}{n} - \frac{\mathrm{B}}{n+1}$$

$$1 = (A)(n+1) - (B)(n);\ \text{Let } n = 0 \Rightarrow A = 1$$

$$\text{Let } n = -1; \Rightarrow B = 1$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$f(n) = \frac{1}{n} \text{ and } f(n+1) = \frac{1}{n+1}$$

$\therefore$ the telescoping series applies and

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = f(1) + \lim_{n\to\infty} \frac{1}{n+1} = f(1) = 1$$

On Excel through 100,852 iterations 0.999990085.

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Example One:

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$$

by Partial Fraction decomposition;

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=\sum\left(\frac{A}{n}-\frac{B}{n+1}\right)=$$

$$\frac{1}{n(n+1)}=\frac{A}{n}-\frac{B}{n+1}$$

$1=(A)(n+1)-(B)(n)$; Let $n=0\Rightarrow A=1$

Let $n=-1;\Rightarrow B=1$

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=\sum\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$f(n)=\frac{1}{n}$ and $f(n+1)=\frac{1}{n+1}$

$\therefore$ the telescoping series applies and

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=f(1)+\lim_{n\to\infty}\frac{1}{n+1}=f(1)=1$$

On Excel through 100,852 iterations 0.999990085.

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Example One.xlsx - Excel

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E1 =SUM(D1:D100852)

	A	B	C	D	E
1	1	1	0.5	0.5	0.999990085
2	2	0.5	0.333333	0.166667	
3	3	0.333333	0.25	0.083333	
4	4	0.25	0.2	0.05	
5	5	0.2	0.166667	0.033333	
6	6	0.166667	0.142857	0.02381	
7	7	0.142857	0.125	0.017857	
8	8	0.125	0.111111	0.013889	
9	9	0.111111	0.1	0.011111	
10	10	0.1	0.090909	0.009091	
11	11	0.090909	0.083333	0.007576	
12	12	0.083333	0.076923	0.00641	
13	13	0.076923	0.071429	0.005495	
14	14	0.071429	0.066667	0.004762	
15	15	0.066667	0.0625	0.004167	
16	16	0.0625	0.058824	0.003676	
17	17	0.058824	0.055556	0.003268	

Sheet1

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Suppose that $g(n) = f(n+k)$.

$$\sum_{n=1}^{\infty}[f(n)-g(n)] =$$
$$f(1)-f(1+k)+$$
$$f(2)-f(2+k)+$$
$$f(3)-f(3+k)+$$
$$\vdots$$
$$f(k)-f(k+k)+$$
$$f(k+1)-f(2k+1)+$$
$$f(k+2)-f(2k+2)+$$
$$f(k+3)-f(2k+3)+$$
$$f(k+4)-f(2k+4)+$$
$$f(k+5)-f(2k+5)+\ldots+$$
$$f(2k)-f(3k)+$$
$$f(2k+1)-f(3k+1)+$$
$$f(2k+2)-f(3k+2)+\ldots$$

Suppose that $g(n) = f(n+k)$.

$$\sum_1^\infty [f(n) - g(n)] =$$

$$f(1) - f(1+k) +$$
$$f(2) - f(2+k) +$$
$$f(3) - f(3+k) + \ldots$$
$$+$$
$$\vdots$$
$$+$$
$$f(k) - f(k+k) +$$
$$f(k+1) - f(2k+1) +$$
$$f(k+2) - f(2k+2) +$$
$$f(k+3) - f(2k+3) +$$
$$f(k+4) - f(2k+4) +$$
$$f(k+5) - f(2k+5) + \ldots +$$
$$\vdots$$
$$f(2k) - f(3k) +$$
$$f(2k+1) - f(3k+1) +$$
$$f(2k+2) - f(3k+2) + \ldots$$
$$\vdots$$
$$f(n-k-3) - f(n-3) +$$
$$f(n-k-2) - f(n-2) +$$
$$f(n-k-1) - f(n-1) +$$
$$f(n-k) - f(n) +$$
$$f(n-k+1) - f(n+1) +$$
$$f(n-k+2) - f(n+2) +$$
$$f(n-k+3) - f(n+3) + \ldots$$

$$f(n-3) - f(n-3+k) +$$
$$f(n-2) - f(n-2+k) +$$
$$f(n-1) - f(n-1+k) +$$
$$f(n) - f(n+k)$$

These are the First terms to cancel

These are Last terms To cancel

Notice that the first k terms of $f(n)$ and the last k terms in the nth partial sums of $g(n)$ do not cancel.

Suppose that $g(n) = f(n+k)$.

$$\sum_{1}^{\infty}[f(n)-g(n)] =$$

$$\begin{array}{l}
f(1)-f(1+k)+ \\
f(2)-f(2+k)+ \\
f(3)-f(3+k)+\ldots \\
+ \\
\vdots \\
+ \\
f(k)-f(k+k)+ \\
f(k+1)-f(2k+1)+ \\
f(k+2)-f(2k+2)+ \\
f(k+3)-f(2k+3)+ \\
f(k+4)-f(2k+4)+ \\
f(k+5)-f(2k+5)+\ldots+ \\
\vdots \\
f(2k)-f(3k)+ \\
f(2k+1)-f(3k+1)+ \\
f(2k+2)-f(3k+2)+\ldots \\
\vdots \\
f(n-k-3)-f(n-3)+ \\
f(n-k-2)-f(n-2)+ \\
f(n-k-1)-f(n-1)+ \\
f(n-k)-f(n)+ \\
f(n-k+1)\quad -f(n+1)+ \\
f(n-k+2)\quad -f(n+2)+ \\
f(n-k+3)\quad -f(n+3)+\ldots\vdots \\
\\
f(n-3)\quad -f(n-3+k)+ \\
f(n-2)\quad -f(n-2+k)+ \\
f(n-1)\quad -f(n-1+k)+ \\
f(n)\quad -f(n+k)
\end{array}$$

These k Terms do **NOT** cancel.

These k Terms do **NOT** cancel

Notice that the first k terms of $f(n)$ and the last k terms in the nth partial sums of $g(n)$ do not cancel.

As mentioned in the introduction all but a finite number of terms cancel.

We can now be more precise and state that the finite number of terms that do not cancel is 2k.

$$\sum_1^{\infty}\left[f(n)-f(n+k)\right]=$$
$$f(1)+f(2)+f(3)+\cdots+f(k)$$
$$-f(n+1)-f(n+2)-f(n+3)-\cdots f(n+k)=$$
$$\sum_{i=1}^{k} f(i)-\lim_{n\to\infty}\sum_{i=1}^{k} f(n+i)$$

$\therefore$ The convergence of the infinite series can be determined by finding $\sum_{i=1}^{k} f(i)-\lim_{n\to\infty}\sum_{i=1}^{k} f(n+i).$

If $\lim_{n\to\infty}\sum_{i=1}^{k} f(n+i)=0$ then the series converges to $\sum_{i=1}^{k} f(i).$

In general if $\lim_{n\to\infty}\sum_{i=1}^{k} f(n+i)\in\Re,$
then the series converges to $\sum_{i=1}^{k} f(i)-\lim_{n\to\infty}\sum_{i=1}^{k} f(n+i).$

If the series is telescoping then the sum of the series can be found by finding the sum of a finite number of terms.

Example Two:
Telescoping series with $k = 1$

$$\sum_{n=1}^{\infty} \frac{10}{4n^2 - 8n + 3} = \sum_{n=1}^{\infty} \left[\frac{5}{2n-3} - \frac{5}{2n-1} \right]$$

$$\sum_{n=1}^{1} \frac{5}{2n-3} - \lim_{n \to \infty} \frac{5}{2n-1} = -5$$

Through 500 iterations on the TI-84 -5.005005005
Through 5000 iterations on the TI-84 -5.00050005

Notice that $f(n) = \dfrac{5}{2n-3}$

and $f(n+1) = \dfrac{5}{2n-1} \therefore k = 1$

Example Three:

Telescoping series with k=2

$$\sum_{n=1}^{\infty} \frac{12}{9n^2+30n+16} = \sum_{n=1}^{\infty} \left[\frac{2}{3n+2} - \frac{2}{3n+8} \right]$$

$$f(n) = \frac{2}{3n+2} \text{ and } g(n) = f(n+2)$$

$$f(n+1) = \frac{2}{3n+5} \to 0 \text{ as n} \to \infty.$$

$$f(n+2) = \frac{2}{3n+8} \to 0 \text{ as n} \to \infty.$$

$$\sum_{n=1}^{2} \frac{2}{3n+2} - \lim_{n\to\infty} \frac{2}{3n+5} - \lim_{n\to\infty} \frac{2}{3n+8} = \frac{13}{20} = .65$$

Through 5000 iterations on Excel .6497334488

Example Four:
Telescoping series transcendental function, $k=1$:

$$\sum_{n=1`}^{\infty}\left[\frac{1}{\ln en}-\frac{1}{\ln(en+e)}\right]=$$

$$f(n)=\frac{1}{\ln en}$$

$$f(n+1)=\frac{1}{\ln(en+e)}$$

$$\sum_{n=1}^{1}\frac{1}{\ln en}-\lim_{n\to\infty}\frac{1}{\ln(en+e)}$$

$$\sum_{n=1}^{1}\frac{1}{\ln en}=1$$

TI-84 through 5000 iterations .894929212
Excel through 102,081 iterations .920214029

Example Five:
Telescoping series transcendental function, $k = 2$,
where $\lim_{n\to\infty} g(n+i) \neq 0$ for some $1 \leq i \leq k$.

$$\sum_{n=0}^{\infty}\left[\arctan n - \arctan(n+2)\right] =$$

$$\sum_{n=0}^{1} \arctan n - \lim_{n\to\infty} \arctan(n+1) - \lim_{n\to\infty} \arctan(n+2) =$$

$$\frac{\pi}{4} - \frac{\pi}{2} - \frac{\pi}{2} = -\frac{3}{4}\pi \approx -2.35619449$$

TI-84 Plus through 5001 iterations = –2.35579461
Excel through 20,322 iterations = – 2.356174669
Note that $f(n) = \arctan n$
$f(n) = \arctan(n)$

$f(n+1) = \arctan(n+1)$

$f(n+2) = \arctan(n+2)$

where $k = 2$

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Example Five:

Telescoping series transcendental function, $k = 2$,

where $\lim_{n\to\infty} g(n+i) \neq 0$ for some $1 \le i \le k$.

$$\sum_{n=0}^{\infty}[\arctan n - \arctan(n+2)] =$$

$$\sum_{n=0}^{1} \arctan n - \lim_{n\to\infty} \arctan(n+1) - \lim_{n\to\infty} \arctan(n+2) =$$

$$\frac{\pi}{4} - \frac{\pi}{2} - \frac{\pi}{2} = -\frac{3}{4}\pi \approx -2.35619449$$

TI-84 Plus through 5001 iterations = −2.35579461

Excel through 100,903 iterations = −2.356174669

Note that $f(n) = \arctan n$

$f(n) = \arctan(n)$

$f(n+1) = \arctan(n+1)$

$f(n+2) = \arctan(n+2)$

where $k = 2$

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Example five.xlsx - Excel

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E1 =SUM(D1:D100903)

	A	B	C	D	E
1	0	0	1.10715	-1.10715	-2.356174669
2	1	0.7854	1.24905	-0.46365	
3	2	1.10715	1.32582	-0.21867	
4	3	1.24905	1.3734	-0.12435	
5	4	1.32582	1.40565	-0.07983	
6	5	1.3734	1.4289	-0.0555	
7	6	1.40565	1.44644	-0.04079	
8	7	1.4289	1.46014	-0.03124	
9	8	1.44644	1.47113	-0.02469	
10	9	1.46014	1.48014	-0.02	
11	10	1.47113	1.48766	-0.01653	
12	11	1.48014	1.49402	-0.01389	
13	12	1.48766	1.49949	-0.01183	
14	13	1.49402	1.50423	-0.0102	
15	14	1.49949	1.50838	-0.00889	
16	15	1.50423	1.51204	-0.00781	
17	16	1.50838	1.5153	-0.00692	
18	17	1.51204	1.51821	-0.00617	

Sheet1

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Example Six:

Telescoping series with $k = 2$

$$\sum_{n=1}^{\infty} \frac{2n^2 + 4n - 2}{n^4 + 4n^3 + 6n^2 + 4n + 5} = \sum_{n=1}^{\infty} \left[\frac{n}{n^2 + 1} - \frac{n+2}{n^2 + 4n + 5} \right]$$

$$f(n) = \frac{n}{n^2 + 1} \text{ and } g(n) = f(n+2)$$

$$f(n+1) = \frac{n+1}{n^2 + 2n + 2} \to 0 \text{ as n} \to \infty.$$

$$f(n+2) = \frac{n+2}{n^2 + 4n + 5} \to 0 \text{ as n} \to \infty.$$

$$\sum_{n=1}^{2} \frac{n}{n^2 + 1} - \lim_{n \to \infty} \frac{n+1}{n^2 + 2n + 2} - \lim_{n \to \infty} \frac{n+2}{n^2 + 4n + 5} =$$

$$\frac{1}{2} + \frac{2}{5} = .9$$

Through 26,658 iterations on Excel = 0.899651

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Example Six:

Telescoping series with $k = 2$

$$\sum_{n=1}^{\infty} \frac{2n^2+4n-2}{n^4+4n^3+6n^2+4n+5} = \sum_{n=1}^{\infty} \left[\frac{n}{n^2+1} - \frac{n+2}{n^2+4n+5} \right]$$

$$f(n) = \frac{n}{n^2+1} \text{ and } g(n) = f(n+2)$$

$$f(n+1) = \frac{n+1}{n^2+2n+2} \to 0 \text{ as n} \to \infty.$$

$$f(n+2) = \frac{n+2}{n^2+4n+5} \to 0 \text{ as n} \to \infty.$$

$$\sum_{n=1}^{2} \frac{n}{n^2+1} - \lim_{n\to\infty} \frac{n+1}{n^2+2n+2} - \lim_{n\to\infty} \frac{n+2}{n^2+4n+5} =$$

$$\frac{1}{2} + \frac{2}{5} = .9$$

Through 26,658 iterations on Excel = 0.899651

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Example6.xlsx - Excel

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K1 =SUM(J1:J26658)

	G	H	I	J	K
1	1	4	20	0.2	0.89992498
2	2	14	85	0.16471	
3	3	28	260	0.10769	
4	4	46	629	0.07313	
5	5	68	1300	0.05231	
6	6	94	2405	0.03909	
7	7	124	4100	0.03024	
8	8	158	6565	0.02407	
9	9	196	10004	0.01959	
10	10	238	14645	0.01625	
11	11	284	20740	0.01369	
12	12	334	28565	0.01169	
13	13	388	38420	0.0101	

Sheet1 Sheet2

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Example Eight:

$$\sum_{n=0}^{\infty}[\arcsin n - \arcsin(n+1)]$$

$$g(n) = f(n+1)$$

but $\arcsin n$ is not defined on Whole numbers.

We are not dealing with an infinite series.

Example Nine:

$$\sum_{n=0}^{\infty}[\sin n - \sin(n+2)]$$

This example appears to be telescoping,

but $\lim_{n\to\infty}\left(\sin n - \sin(n+2)\right) = \lim_{n\to\infty}\left[-2(\sin 1)\cos(n+1)\right]$

DNE and $\therefore$ does not converge.

Example 10:

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^4+2n^3+3n^2+2n+2} =$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{n^2+1} - \frac{1}{(n+1)^2+1}\right]$$

$g(n) = f(n+1)$ and satisfies telescoping requirements.

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^4+2n^3+3n^2+2n+2} =$$

$$\sum_{n=1}^{1} \frac{1}{x^2+1} - \lim_{n\to\infty} \frac{1}{(n+1)^2+1} = .5$$

Excel through 25,814 iterations 0.499999998

MathType - Equation in The Telescoping Series Extended ACTIVE Revised

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Example 10:

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^4+2n^3+3n^2+2n+2} =$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{n^2+1} - \frac{1}{(n+1)^2+1}\right]$$

$g(n) = f(n+1)$ and satisfies telescoping requirements.

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^4+2n^3+3n^2+2n+2} =$$

$$\sum_{n=1}^{1} \frac{1}{x^2+1} - \lim_{n\to\infty} \frac{1}{(n+1)^2+1} = .5$$

Excel through 25,814 iterations 0.499999998

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Example 10.xlsx - Excel

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E1 =SUM(D1:D25814)

	A	B	C	D	E
1	1	3	10	0.3	0.499999998
2	2	5	50	0.1	
3	3	7	170	0.041176	
4	4	9	442	0.020362	
5	5	11	962	0.011435	
6	6	13	1850	0.007027	
7	7	15	3250	0.004615	
8	8	17	5330	0.003189	
9	9	19	8282	0.002294	
10	10	21	12322	0.001704	
11	11	23	17690	0.0013	
12	12	25	24650	0.001014	
13	13	27	33490	0.000806	
14	14	29	44522	0.000651	
15	15	31	58082	0.000534	
16	16	33	74530	0.000443	
17	17	35	94250	0.000371	

Sheet1

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Example 11

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+2}-\sqrt{n+1}}{\sqrt{n^2+3n+2}} =$$

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+2}-\sqrt{n+1}}{\sqrt{(n+1)(n+2)}} =$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{\sqrt{(n+1)}} - \frac{1}{\sqrt{(n+2)}}\right]$$

Satisfies telescoping requirements with $k=1$

$$\sum_{n=1}^{1} \frac{1}{\sqrt{(n+1)}} - \lim_{n\to\infty} \frac{1}{\sqrt{(n+2)}} = \frac{\sqrt{2}}{2} \approx .7071067812$$

Excel through 113,247 iterations 0.704135

MathType - Equation in The Telescoping Series Extended ACTIVE Revised

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Example 11

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+2}-\sqrt{n+1}}{\sqrt{n^2+3n+2}} =$$

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+2}-\sqrt{n+1}}{\sqrt{(n+1)(n+2)}} =$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{\sqrt{(n+1)}} - \frac{1}{\sqrt{(n+2)}}\right]$$

Satisfies telescoping requirements with $k = 1$

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Excel through 113,247 iterations 0.704135

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Example 11.xlsx - Excel

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F3

	A	B	C	D	E	F
1	1	1.732051	1.414214	2.44949	0.129757	0.704135
2	2	2	1.732051	3.464102	0.07735	
3	3	2.236068	2	4.472136	0.052786	
4	4	2.44949	2.236068	5.477226	0.038965	
5	5	2.645751	2.44949	6.480741	0.030284	
6	6	2.828427	2.645751	7.483315	0.024411	
7	7	3	2.828427	8.485281	0.02022	
8	8	3.162278	3	9.486833	0.017106	
9	9	3.316625	3.162278	10.48809	0.014716	
10	10	3.464102	3.316625	11.48913	0.012836	
11	11	3.605551	3.464102	12.49	0.011325	
12	12	3.741657	3.605551	13.49074	0.010089	
13	13	3.872983	3.741657	14.49138	0.009062	
14	14	4	3.872983	15.49193	0.008199	
15	15	4.123106	4	16.49242	0.007464	

Sheet1

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Summary:

The Telescoping series is an effective technique for finding the sum of an infinite series and can be used when the series is of the form

$$\sum_{n=1}^{\infty}[f(n)-f(n+k)]$$

where k is a natural number.

It can be used on rational, algebraic and transcendental functions.

Notice that if an infinite series is telescoping then not only can convergence be determined, but the sum of the series can also be found.

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