2017 HAWAII UNIVERSITY INTERNATIONAL CONFERENCES
SCIENCE, TECHNOLOGY & ENGINEERING, ARTS, MATHEMATICS & EDUCATION JUNE 8 - 10, 2017
HAWAII PRINCE HOTEL WAIKIKI, HONOLULU, HAWAII

AN ANALYSIS OF CAPITAL F IN THE FUNDAMENTAL THEOREM OF CALCULUS

GARRISON, JOSEPH M.
DEPARTMENT OF MATHEMATICS
MIDDLE GEORGIA UNIVERSITY
COCHRAN, GEORGIA.

Prof. Joseph M Garrison
Department of Mathematics
Middle Georgia State University
Cochran, Georgia.

An Analysis of Capital F in the Fundamental Theorem of Calculus

Synopsis:

This paper will reflect on the integrating function central to the statement of the theorem which will be called Capital F. Capital F's relationship to all of the other antiderivatives of f(x) and the various structures of capital F producing the same function will be examined.

An analysis of Capital F in the Fundamental Theorem of Calculus

<u>The Fundamental Theorem of Calculus</u>

Suppose $\int_c^x f(t)dt$ exists for all $x, c \in [a,b]$; c is a constant real number.

Then if $F_c(x) = \int_c^x f(t)dt$, it follows that

$$\frac{d}{dx} F_c(x) = \frac{d}{dx} \int_c^x f(t)\, dt = f(x) \forall \; x, \; c \in [a,b]; \text{ c is a constant real number.}$$

After eluding Archimedes, Descartes, Barrow, Fermat and others (https://en.wikipedia.org/wiki/History_of_calculus) the fundamental theorem of calculus was finally discovered and validated through differentials and fluxions by Isaac Newton and later rigorously proved using limits by Augustin Louis Cauchy (1789--1857). “The modern proof of the Fundamental Theorem of Calculus was written in his Lessons Given at the École Royale Polytechnique on the Infinitesimal Calculus in 1823. Cauchy's proof finally rigorously and elegantly united the two major branches of calculus (differential and integral) into one structure.”

https://www3.nd.edu/~apilking/Math10550/Lectures/26.%20Fundamental%20Theorem.pdf

This paper will reflect on the integrating function, which is central to the statement of the fundamental theorem of calculus. We will call this function Capital F. Capital F’s relationship to all of the other antiderivatives of f(x) and the various structures of capital F producing the same function will be examined.

Proposition One:

In other words, can all the antiderivatives of f(x) be generated by an appropriate choice of c in

$\int_c^x f(t)dt$ where $x,c \in \Re$?

This paper will show that in many cases it is not possible to generate all the antiderivatives of f(x) by an appropriate choice of c.

Note that we are not arguing that a given antiderivative does not exist, but that it cannot be generated by choosing a real number for the lower index

of the given $F_c(x) = \int_c^x f(t)\,dt$ which we are calling Capital F.

Proposition Two:

Suppose $F_c(x) = \int_c^x f(t)dt$ and $f(x)$ is continuous on $\Re$; c is a constant.

Does there exist $c_1 \neq c \in \Re$ such that

$$F_c(x) = \int_c^x f(t)dt = F_{c_1}(x) = \int_{c_1}^x f(t)dt?$$

In other words, are there different values of c, which produce the same F?

This paper will show that quite often there are many different

values of c producing the same Capital F.

These values occur at the real zeros of Capital F!

Proposition One analyzed, first glance:

Can all the antiderivatives of f$\left(\mathrm{x}\right)$ be generated by an appropriate choice of c in $\int_c^x f(t)dt$ where x,c $\in \Re$?

Let $F_c\left(x\right)=\int_c^x f\left(t\right)dt$ for a given c $\in$ $\Re$.

Now consider an antiderivative of $f\left(x\right)$, say $G\left(x\right)$,

obtained by choosing an appropriate k_1 so that $G\left(x\right)=F_c\left(x\right)+k_1$

and $G\left(x\right)$ has no real roots. This k_1 exists when $F_c\left(x\right)$ has either an upper bound or a lower bound.

Without loss of generality assume l is a lower bound.

Choose $k_1>\left|l\right|$.

Since $F_c\left(x\right)\geq l$,

$F_c\left(x\right)+k_1>l+\left|l\right|>0\Rightarrow F_c\left(x\right)+k_1>0$.

Then G(x) cannot be represented by

$G_{c_2}\left(x\right)=\int_{c_2}^x f\left(t\right)dt$, $\mathrm{c}_2\in\Re$ because $G_{c_2}\left(c_2\right)=0\Rightarrow c_2$ is a real root.

But G(x) has no real root on $\Re$.

Analysis Continued:

Let $F_0(x) = \int_0^x t\,dt$ be a given antiderivative of $f(x) = x$.

$F_0(x) = \frac{1}{2}x^2$ because $F_0(x) = \int_0^x t\,dt = \frac{1}{2}t^2 \Big|_0^x = \frac{1}{2}x^2$

(fundamental theorem of calculus).

Now any other antiderivative, say G(x), can be

represented by $G(x) = F_0(x) + k = \frac{1}{2}x^2 + k$ (mean value theorem).

Now what value of c in $F_c(x) = \int_c^x t\,dt$

would produce the derivative $\frac{1}{2}x^2 + k$?

Since $\frac{1}{2}x^2 + k = F_c(x) = \int_c^x t\,dt = \frac{1}{2}x^2 - \frac{1}{2}c^2$,

then $-\frac{1}{2}c^2 = k \Rightarrow c^2 = -2k$

$\Rightarrow k \le 0$. Therefore no antiderivative

of the form $y = \frac{1}{2}x^2 + k; k \in \Re^+$

can be represented by $\int_c^x t\,dt$.

Example for proposition One:

Suppose $f(x) = x$.

$G(x) = y = \frac{1}{2}x^2 + 4$ is a specific antiderivative for $f(x)$.

Let $F(x) = \int_c^x t\,dt$

What real value for c above would generate $G(x)$?

$F(x) = \int_c^x t\,dt$ is one antiderivative (fundamental theorem of calculus)

Any other antiderivative of x can be represented by

$\frac{1}{2}x^2 + k$ where k is a constant real number(mean value theorem).

$F(x) = \int_c^x t\,dt = \frac{1}{2}t^2\Big|_c^x = \frac{1}{2}x^2 - \frac{1}{2}c^2 = \frac{1}{2}x^2 + 4$

The question becomes whether there exists $c \in \Re$ such that

$-\frac{1}{2}c^2 = 4$ which would require that $c^2 = -8$ and then $c \notin \Re$.

Any antiderivative of the form $\frac{1}{2}x^2 + k$

must have $k \le 0$ to be represented by $F(x) = \int_c^x t\,dt$.

Therefore, any antiderivative of the form $\frac{1}{2}x^2 + k$ where k is positive

cannot be represented by $F(x) = \int_c^x t\,dt$.

Let $F_{c_i}(x)=\int_{c_i}^{x} f(t)dt$ be a given antiderivative of $f(x)$ that is continuous on $\Re$.

$F_{c_i}(x)=\int_{c_i}^{x} f(t)dt$ exists by the fundamental theorem of calculus.

Now any other antiderivative, say G(x), can be

represented by $G(x)=F_{c_i}(x)+k=\int_{c_i}^{x} f(t)dt+\mathrm{k}$ (mean value theorem).

$F_{c_i}(x)=\int_{c_i}^{x} f(t)dt \ =F_{c_i}(x)-F_{c_i}(c_i)$.

Now what value of c in $F_c(x)=\int_{c}^{x} f(t)\,dt$

would produce the antiderivative $G(x)=F_{c_i}(x)+k$?

$G(x)=F_{c_i}(x)+k=\int_{c}^{x} f(t)\,dt=F_{c_i}(x)-F_{c_i}(c)\Rightarrow$

$-F_{c_i}(c)=k$

and $F_{c_i}(c)+k=0=G(c)$

$\Rightarrow$ that for the antiderivative $G(x)=F_{c_i}(x)+k$

to be represented by $\int_{c}^{x} f(t)\,dt$, then the set containing the real roots

of $G(x)$ must not be empty; and furthermore

all elements in this set produce

identical $F_{c_i}(x)=\int_{c_i}^{x} f(t)dt$.

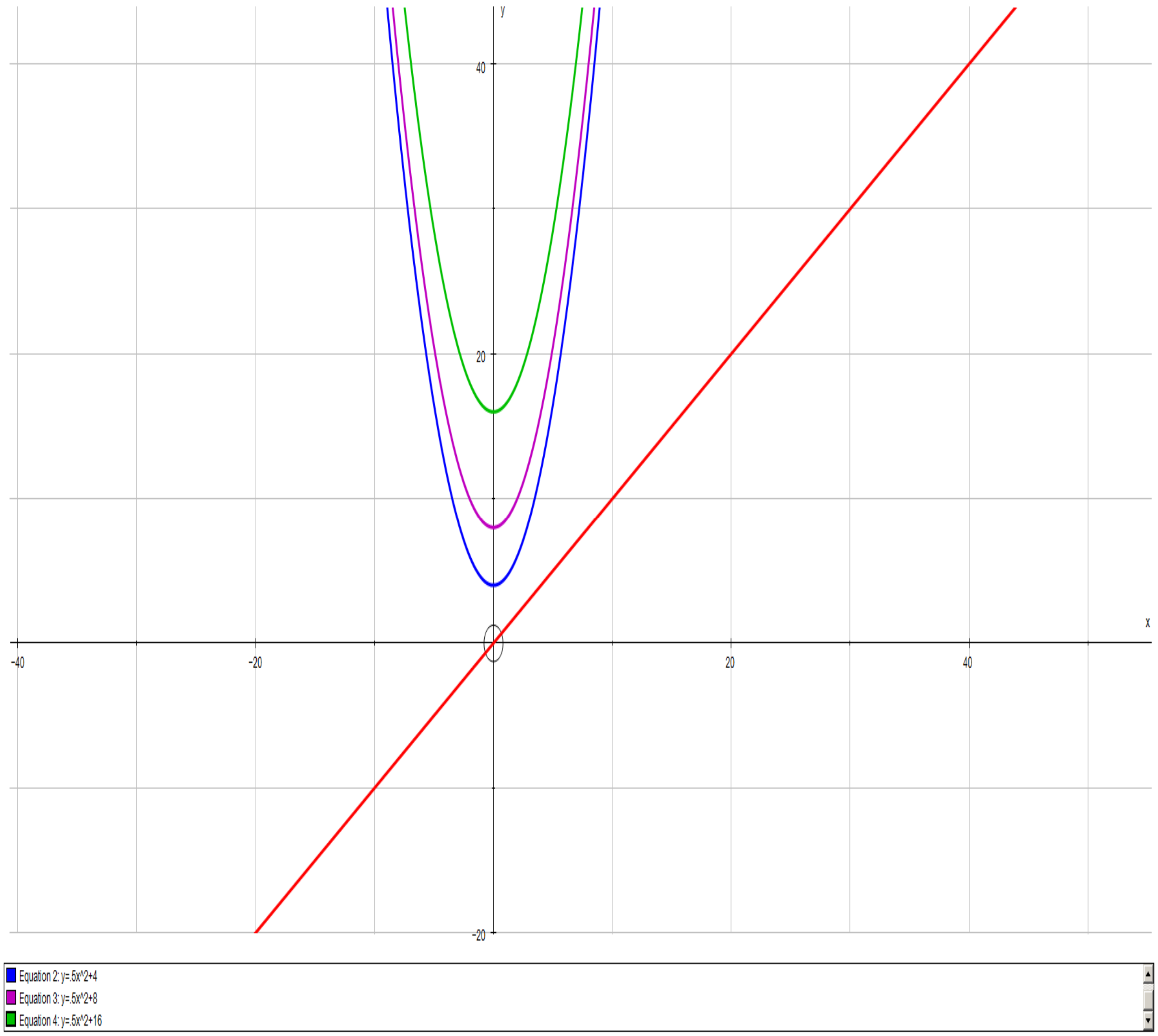
More Examples for Proposition One:

Antiderivatives with no real lower index in the generating function $F(x)=\int_c^x t\ dt$:

$$F_1(x)=\frac{1}{2}x^2+1$$

$$F_2(x)=\frac{1}{2}x^2+4$$

$$F_3(x)=\frac{1}{2}x^2+16$$



Example:

Antiderivatives with real lower index in the generating function $F(x) = \int_c^x t\ dt$:

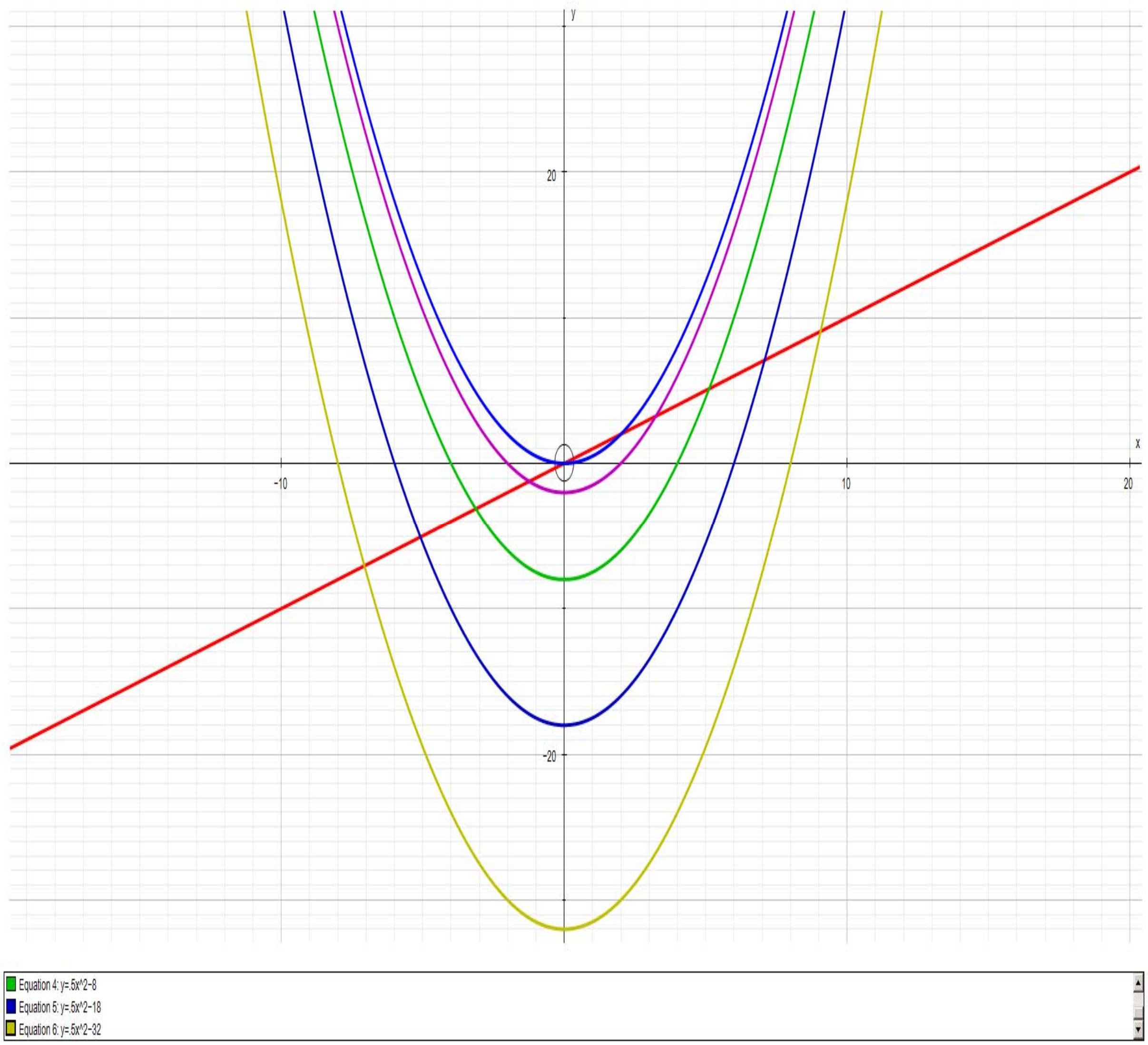
$F_1(x) = \frac{1}{2}x^2$ when c=0

$F_2(x) = \frac{1}{2}x^2 - 2$ when $c = \pm 2$

$F_3(x) = \frac{1}{2}x^2 - 8$ when $c = \pm 4$

$F_4(x) = \frac{1}{2}x^2 - 18$ when $c = \pm 6$

$F_5(x) = \frac{1}{2}x^2 - 32$ when $c = \pm 8$



Proposition Two analyzed:

Suppose $c_1 \neq c$ and $\int_c^x f(t)dt = \int_{c_1}^x f(t)dt.$

$$\int_c^x f(t)dt - \int_{c_1}^x f(t)dt = 0$$

$$\int_c^x f(t)dt + \int_x^{c_1} f(t)dt = 0$$

$$\int_c^{c_1} f(t)dt = 0 \Rightarrow$$

c_1 is a root of $\int_c^x f(t)dt.$

We also know that c is a root of $\int_c^x f(t)dt;$

therefore the antiderivatives are = when the $c_1 's$ are roots.

Suppose $F_{c_1}(x) = \int_{c_1}^x f(t)\,dt.$

Suppose c_2 is a root of $F_{c_1}(x), c_1 \neq c_2$

(note that c_1 is a also root of $F_{c_1}(x)$).

Then $\int_{c_1}^{c_1} f(t)\,dt = 0$

and $F_{c_1}(c_2) = \int_{c_1}^{c_2} f(t)\,dt = 0$ because c_2 is a root.

Consider $x_1 \in$ the open interval $\left(c_1, c_2\right)$.

$$\int_{c_1}^{c_2} f\left(x\right)dx = \int_{c_1}^{x_1} f\left(x\right)dx + \int_{x_1}^{c_2} f\left(x\right)dx = 0$$

or $\int_{c_1}^{x_1} f\left(x\right)dx = -\int_{x_1}^{c_2} f\left(x\right)dx = \int_{c_2}^{x_1} f\left(x\right)dx \Rightarrow$

$F_{c_1}(x_1) = F_{c_2}(x_1);$

therefore $F_{c_1}(x) = F_{c_2}(x),$ and there are two distinct values of c producing the same Capital F.

Example for Proposition Two:

Suppose $F(x)=\int_1^x (2t-3)dt.$

Then $F(1)=0,$ and

all antiderivatives are of the form $y=x^2-3x+c.$

$F(1)=0=1-3+c \Rightarrow c=2,$

so $F(x)=x^2-3x+2.$

Note that f(2)=0.

$F(x)=\int_1^x (2t-3)dt = F(x)=\int_2^x (2t-3)dt.$

Thus, we have two values for the lower limit of integration producing the same F(x).

These lower limits always occur at the real roots of $F(x)=x^2-3x+2$.

The Connection between Proposition I and Proposition II:

Suppose there exists a real c such that

$F_c(x) = \int_c^x f(t)dt$. $F_c(x) = \int_c^x f(t)dt$ is an antiderivative of $f(x)$,

continuous on $\Re$(fundamental theorem of calculus).

Now any other antiderivative, say G(x), can be

represented by $G(x) = F_c(x) + k$(mean value theorem).

What value of c_1 in $F_{c_1}(x) = \int_{c_1}^x f(t)\,dt$

would produce the antiderivative $G(x) = F_c(x) + k$?

$G(x) = F_c(x) + k = \int_c^x f(t)\,dt + k = F_{c_1}(x) = \int_{c_1}^x f(t)dt = F_c(x) - F_c(c_1)$

$F_c(x) - F_c(c) + k = F_c(x) - F_c(c_1) \Rightarrow$

$F_c(x) - 0 + k = F_c(x) - F_c(c_1)$

and $F_c(c_1) = -k$.

Thus for the antiderivative $G(x) = F_c(x) + k$

to be represented by $\int_{c_1}^x f(t)\,dt$, the set containing the roots

of $F_c(c_1) = -k$, or $G(x) = 0$ must not be empty; and furthermore,

all elements in this set are real numbers producing

identical functions $F_c(x) = \int_c^x f(t)\,dt$.

Given an antiderivative, $F_c(x) = \int_c^x f(t)dt$,of a continuous function and a desired constant of integration, k, in another antiderivative, in order to determine if the new antiderivative can be written in terms of Capital F with a real c solve the equation $F_c(x) + k = \int_{c_1}^x f(t)dt$ =0

$$\text{or solve } G(x) = 0.$$

If there are real roots, it can be written in terms of Capital F with a real c.
The real roots all produce equal Capital F’s.

Example:

Suppose $F_c(x) = \int_c^x (2t+1)\,dt.$

$F_0(x) = \int_0^x (2t+1)\,dt =$

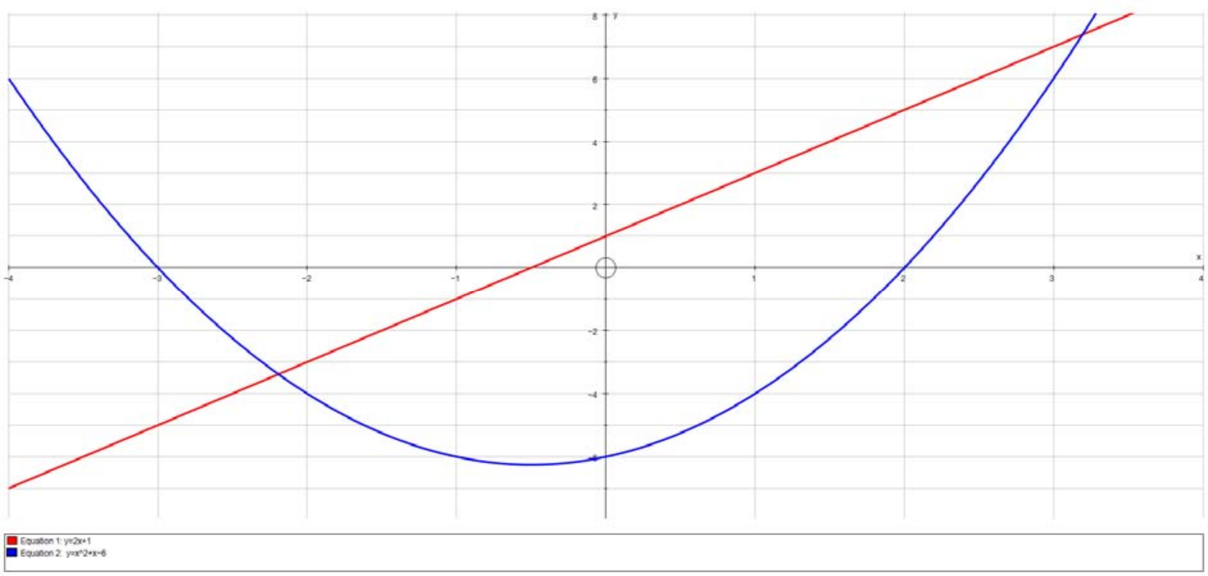
$(t^2+t)\big|_0^x = x^2 + x$ (fundamental theorem).

Analyze $x^2 + x - 6 = 0$

The roots $\{-3, 2\}$ are values for c that

produce the same $F_c(x) = \int_c^x (2t+1)\,dt = x^2 + x - 6.$

$$F_{2,-3}(x) = \int_2^x (2t+1)\,dt = \int_{-3}^x (2t+1)\,dt = x^2 + x - 6$$



Example:

Suppose $F_c(x) = e^x + k$.

What values of k allow $F_c(x) = e^x + k$ to be written as $\int_c^x e^t\, dt$?

Then $e^x + k = 0$ when $k < 0$.

Concluding Example

Note that any polynomial of odd degree can be represented by $\int_c^x f(t)dt$.

Suppose G(x) $= x^3 - 2x^2 - x + 2$.

What values of c allow G(x) to be written as $F(x) = \int_c^x \left(3t^2 - 4t - 1\right) dt$?

Solve G(x) $= x^3 - 2x^2 - x + 2 = 0$.

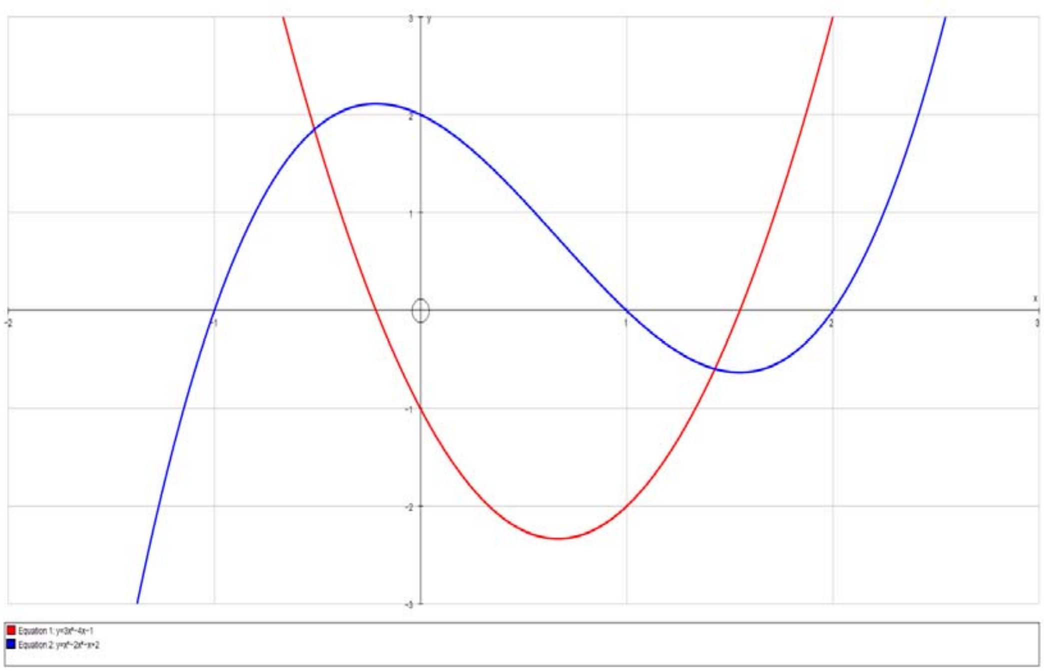
The roots are $\{-1,\ 1,\ 2\}$.

$$G(x) = x^3 - 2x^2 - x + 2 =$$

$$\int_{-1}^{x} \left(3t^2 - 4t - 1\right) dt =$$

$$\int_{1}^{x} \left(3t^2 - 4t - 1\right) dt =$$

$$\int_{2}^{x} \left(3t^2 - 4t - 1\right) dt.$$



Conclusion:

Let f(x) be the derivative of a differentiable function, say G(x). The integrating functions $F_{c_i}(x) = \int_{c_i}^{x} f(t)dt$ used in the Fundamental Theorem are equal when the c_i's are real roots of G(x), and $F_{c_i}(x) = \int_{c_i}^{x} f(t)dt = G(x)$. There are many times when an antiderivative of f(x) cannot be generated by $c_i \in \Re$ in $F(x) = \int_{c_i}^{x} f(t)dt$. This occurs when G(x) has no real root. The lower index c_i determines the value of the constant of integration k, in the antiderivative $F(x) = G(x) + k$ where $G(x)$ is any antiderivative because c_i is a root of $F(x)$. The final concern is whether there is a representation of G(x) $= F_{c_i}(x) = \int_{c_i}^{x} f(t)dt$ where f(x) is not the derivative of G(x). The fundamental theorem of calculus guarantees that no such function exists.

QED