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ON THE INFINITE DIVISIBILITY OF PROBABILITY DISTRIBUTIONS WITH DENSITY FUNCTION OF NORMED PRODUCT OF CAUCHY DENSITIES

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On the Infinite Divisibility of Probability Distributions with Density Function of Normed Product of Cauchy Densities

Synopsis:

The speaker will talk about the possible infinite divisibility of probability distributions with density function of normd product of the multi-dimensional Cauchy densities by using the soft "Mathematica“.

ON THE INFINITE DIVISIBILITY OF PROBABILITY DISTRIBUTIONS WITH DENSITY FUNCTION OF NORMED PPRODUCT OF THE CAUCHY DENSITIES

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1 Introduction

It is well-known that the Cauchy distribution is infinitely divisible. This fact is coming from the following equality,

$$\begin{aligned} \int_{-\infty}^{\infty} e^{itx}\frac{1}{\pi(1+x^2)}dx &= e^{-|t|} \\ &= (e^{-\frac{|t|}{n}})^n \\ &= \Big(\int_{-\infty}^{\infty} e^{ity}\frac{\frac{1}{n}}{\pi(\frac{1}{n^2}+y^2)}dy\Big)^n \end{aligned}$$

for every positive integer n. This shows the Cauchy distribution can be expressed as the $(n-1)$-fold convolutions of the Cauchy distribution with itself. It is known that all of the probability distribution with the density function of normed product of the n Cauchy densities

$$f(1,2,\cdots,n;x) = \frac{c}{(1+x^2)(2^2+x^2)(3^2+x^2)\cdots(n^2+x^2)}$$

are infinitely divisible, and it is also known that the density function of normed product of two 3 dimensional Cauchy densities

$$f(1,2;x) = \frac{c}{(1+|x|^2)^2(2^2+|x|^2)^2}, \quad x=(x_1,x_2,x_3)\in \mathbf{R}^3$$

is infinitely divisible. But it seems that it is not known if the probability distriution with the density function of normed product of triple 3 dimensional Cauchy densities is infinitely divisible or not. The goal of this note is to obtain a prospect that the following probability density

$$f(1,2,3,\cdots,n;x) = \frac{c}{(1+|x|^2)^2(2^2+|x|^2)^2(3^2+|x|^2)^2\cdots(n^2+|x|^2)^2},$$

$$x = (x_1, x_2, x_3) \in \mathbf{R}^3$$

is infinitely divisible. In this note, the author discusses on the infinite divisiblity of the probability distribution with the density function of normed product of the triple 3 dimensional Cauchy densities such as

$$f(1,2,3;x) = \frac{c}{(1+|x|^2)^2(2^2+|x|^2)^2(3^2+|x|^2)^2}, \tag{1}$$

where

$$x = (x_1, x_2, x_3) \in \mathbf{R}^3$$

and c is determined by

$$\int_{\mathbf{R^3}} f(1,2,3;x)dx = 1.$$

We give the definition of an infinitely divisible distribution in the 1-dimensional case. For the definition of an infinitely divisible distribution in the multi-dimensional case, refer to Sato's book [7]. A probability distribution function $F(x)$ is called an infinitely divisible distribution if for each integer $n > 1$ there is a probability distribution $F_n(x)$ such that

$$F(x) = (F_n * \cdots * F_n)(x),$$

(* denotes the convolution.). The convolution is defined by

$$(F_n * F_n)(x) = \int_{-\infty}^{\infty} F_n(x-y)dF_n(y)$$

In terms of random variable we shall say that the random variable X is infinitely divisible if for every natural number n it can be represented as the sum $X = X_{n1} + \cdots + X_{nn}$ of n independent identically distributed random variables $X_{1n,\ldots,}X_{nn}$and $F_n(x)$ is the probability distribution function $P(X_{nj} \le x)$. If a probability distribution function $F(x)$ is 0 over the interval $(-\infty, 0)$ and infinitely divisible, and if we denote the Laplace-Stieltjes taransforms of the probability distributions $F(x)$ and $F_n(x)$,

$$L(s) = \int_0^{\infty} e^{-sx}dF(x), \quad L_n(s) = \int_0^{\infty} e^{-sx}dF_n(x),$$

the equality

$$L(s) = (L_n(s))^n$$

holds. It is known that the Laplace-Stieltjes transform of an infinitely divisible probability distribution $F(x)$ over the interval $[0, \infty)$ can be written as follows:

$$L(s) = \exp\{\int_{-0}^{\infty} (e^{-sx} - 1)\frac{1}{x} dK(x)\}$$

where

(c1) $K(x)$ is nondecreasing,

(c2) $K(-0) = 0$,

(c3) $\int_1^\infty 1/x \, dK(x) < \infty$.

An infinitely divisible probability distribution $F(x)$ on $[0, \infty)$ satisfies an integral equation,

$$\int_0^x t \, dF(t) = \int_0^x F(x-t) dK(t), \; x > 0$$

and in particular if the probability distribution function $F(x)$ on $[0, \infty)$ has a density function $f(x)$, the density funcion $f(x)$ satisfies the following integral equation:

$$xf(x) = \int_0^x f(x-t) dK(t), \; x > 0$$

(cf. F. Steutel [8]).
If $dK(t)$ is absolutely continuous, i.e. $dK(t) = k(t)dt$ we obtain the integral equation

$$xf(x) = \int_0^x f(x-t)k(t)t, \; x > 0.$$

Making use of the Laplace transform we obtain

$$-L'(s) = L(s)L(k; s),$$

where

$$L(s) = \int_0^\infty \exp\{-sx\} f(x) dx,$$

$$L(k; s) = \int_0^\infty \exp\{-st\} k(t) dx.$$

And we can find the spectral function $k(t)$ by making use of the inverse Laplace transform on the figure after the references. By computations the author tries to show that the spectral function $k(t)$ for the mixing density $g(3; 3; v)$ is a positive function.

2 The variance mixture density of the Normal distributions

Consider the probability density function with normed product of the triple 3 dimensional Cauchy densities

$$
\begin{aligned}
& f(a_1, a_2, a_3; x) \\
= \quad & \frac{c}{(a_1^2 + |x|^2)^{(d+1)/2}(a_2^2 + |x|^2)^{(d+1)/2}(a_3^2 + |x|^2)^{(d+1)/2}}
\end{aligned}
$$

$d = 3;\ a_1 = 1,\ a_2 = 2,\ a_3 = 3$. The number c is normalized such that

$$\int_{R^3} f(x)dx = 1.$$

We have

$$\frac{\Gamma((d+1)/2)}{(a^2 + |x|^2)^{(d+1)/2}} = \int_0^\infty \exp\{-t(a^2 + |x|^2)\} \cdot t^{(d+1)/2-1} dt$$

and see that

$$
\begin{aligned}
& f(x) = \frac{c}{\{\Gamma((d+1)/2)\}^3} \int_0^\infty \exp\{-t_1(a_1^2 + |x|^2)\} \cdot t_1^{(d+1)/2-1} dt_1 \\
& \int_0^\infty \exp\{-t_2(a_2^2 + |x|^2)\} \cdot t_2^{(d+1)/2-1} dt_2 \\
& \int_0^\infty \exp\{-t_3(a_3^2 + |x|^2)\} \cdot t_3^{(d+1)/2-1} dt_3 \\
= \quad & \frac{c}{\{\Gamma((d+1)/2)\}^3} \int_0^\infty \int_0^\infty \int_0^\infty \exp\{-(t_1 + t_2 + t_3)|x|^2 \\
- \quad & a_1^2 t_1 - a_2^2 t_2 - a_3^2 t_3\} \cdot (t_1 t_2 t_3)^{(d+1)/2-1} dt_1 dt_2 dt_3
\end{aligned}
$$

By a change of variables,

$$t_1 = u_1,\ t_2 = u_2,\ t_3 = u_3 - u_1 - u_2$$

we have the Jacobian,

$$\frac{D(t_1, t_2, t_3)}{D(u_1, u_2, u_3)} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{vmatrix} = 1$$

and we have

$$
\begin{aligned}
& f(x) = \frac{c}{\{\Gamma((d+1)/2)\}^3} \int \int \int_{0 \le u_1, u_2, u_3; u_1 + u_2 \le u_3} \exp\{-u_3|x|^2 \\
- \quad & a_1^2 u_1 - a_2^2 u_2 - a_3^2 (u_3 - u_1 - u_2)\} \\
\cdot \quad & \{u_1 u_2 (u_3 - u_1 - u_3)\}^{(d+1)/2-1} du_1 du_2 du_3
\end{aligned}
$$

If $d = 3$ then $(d+1)/2 = 2$ and $\Gamma(2) = 1$. From the above we see that

$$\begin{aligned} f(x) &= c\int\int\int_{0\le u_1,u_2,u_3;u_1+u_2\le u_3} \exp\{-u_3|x|^2 \\ - &\ a_1^2u_1 - a_2^2u_2 - a_3^2(u_3-u_1-u_2)\} \\ \cdot &\ \{u_1u_2(u_3-u_1-u_3)\}du_1du_2du_3. \end{aligned}$$

Making use of the Fubini theorem since the integrand is non-negative, we can change the order of the three fold integrals such as

$$\begin{aligned} f(x) &= c\int_0^\infty \exp\{-u_3|x|^2\} \\ \cdot &\ \Big[\exp\{-a_3^2u_3\}\int\int_{0\le u_1,u_2;u_1+u_2\le u_3} \\ &\ \exp\{(a_3^2-a_1^2)u_1 + (a_3^2-a_2^2)u_2\} \\ \cdot &\ \{u_1u_2(u_3-u_1-u_2)\}du_1du_2\Big]du_3. \end{aligned}$$

Make a change of variables, $u_1 = u_3v_1, u_2 = u_3v_2$, and ignore the boundary set of the volume measure zero in the above three fold integrals $\{(u_1,u_2,u_3) : u_1 = 0, 0 \le u_2 \le u_3\} \cup \{(u_1,u_2,u_3) : u_2 = 0, 0 \le u_1 \le u_3\} \cup \{0 \le u_1, u_2; u_1 + u_2 = u_3\}$. Then we have

$$\begin{aligned} f(x) &= c\int_0^\infty \exp\{-u_3|x|^2\} \\ \cdot &\ \Big[\exp\{-a_3^2u_3\}\int\int_{0<v_1,v_2;v_1+v_2<1} \\ &\ \exp\{(a_3^2-a_1^2)u_3v_1 + (a_3^2-a_2^2)u_3v_2\} \\ \cdot &\ \{u_3v_1u_3v_2(u_3-u_3v_1-u_3v_2)\}u_3^2dv_1dv_2\Big]du_3 \\ = &\ c\int_0^\infty \exp\{-u_3|x|^2\} \\ \cdot &\ \Big[\exp\{-a_3^2u_3\}\int\int_{0<v_1,v_2;v_1+v_2<1} \\ &\ \exp\{((a_3^2-a_1^2)v_1 + (a_3^2-a_2^2)v_2)u_3\} \\ \cdot &\ \{v_1v_2(1-v_1-v_2)\}dv_1dv_2\Big]u_3^5du_3. \end{aligned}$$

Make a change of a variable, $u_3 = 1/v$ and we obtain

$$\begin{aligned} f(x) &= \int_0^\infty \frac{1}{v^2}\exp\{-\frac{|x|^2}{v}\} \\ \cdot &\ \Big[\frac{c}{v^5}\exp\{-a_3^2/v\}\int\int_{0<v_1,v_2;v_1+v_2<1} \\ &\ \exp\{((a_3^2-a_1^2)v_1 + (a_3^2-a_2^2)v_2)/v\} \\ \cdot &\ \{v_1v_2(1-v_1-v_2)\}dv_1dv_2\Big]dv \end{aligned}$$

and we see that

$$
\begin{aligned}
f(x) = & \int_0^\infty \frac{1}{(\pi v)^{3/2}} \exp\{-\frac{|x|^2}{v}\} dy \Big[\frac{c\pi^{3/2}}{v^{11/2}} \exp\{-\frac{a_3^2}{v}\} \\
\cdot & \int\int_{0<v_1, v_2; v_1+v_2<1} \exp\{((a_3^2 - a_1^2)v_1 + (a_3^2 - a_2^2)v_2)/v\} \\
\cdot & \{v_1 v_2 (1 - v_1 - v_2)\} dv_1 dv_2 \Big].
\end{aligned}
$$

We obtain the variance mixture density function (1) of the 3 dimensional normal density functions. The mixing density function is

$$
\begin{aligned}
g(3;3;v) = & \frac{c\pi^{3/2}}{v^{11/2}} \exp\{-\frac{a_3^2}{v}\} \\
\cdot & \int\int_{0<v_1, v_2; v_1+v_2<1} \exp\{((a_3^2 - a_1^2)v_1 + (a_3^2 - a_2^2)v_2)/v\} \\
\cdot & \{v_1 v_2 (1 - v_1 - v_2)\} dv_1 dv_2.
\end{aligned}
$$

If this probability density function is infinitely divisible then the probability density function (1) is infinitely divisible (cf. Feller [3]). The author is going to show that the mixing density function $g(3;3;v)$ is possible infinitely divisible.

3 The mixing density function $g(3;3;v)$

By a change of variables in the above double integrals

$$
v_1 = u_1,\ v_2 = (1 - u_1)y
$$

we have the Jacobian,

$$
\frac{D(v_1, v_2)}{D(u_1, y)} = \begin{vmatrix} 1 & 0 \\ -y & 1 - u_1 \end{vmatrix} = 1 - u_1.
$$

We see that

$$
\begin{aligned}
& g(3;3;v) = \frac{c\pi^{3/2}}{v^{11/2}} \exp\{-\frac{a_3^2}{v}\} \\
\cdot & \int\int_{0<v_1<1; 0<y<1} \exp\{((a_3^2 - a_1^2)v_1 + (a_3^2 - a_2^2)(1 - v_1)y)\frac{1}{v}\} \\
& \Big[v_1(1 - v_1)^2 y(1 - y)\Big](1 - v_1) dy dv_1 \\
= & \frac{c\pi^{3/2}}{v^{11/2}} \exp\{-\frac{a_3^2}{v}\} \\
\cdot & \int\int_{0<v_1<1; 0<y<1} \exp\{((a_3^2 - a_1^2)v_1 + (a_3^2 - a_2^2)(1 - v_1)y)\frac{1}{v}\} \\
& v_1(1 - v_1)^3 y(1 - y) dv_1 dy.
\end{aligned}
$$

At last, we obtain a simple form of the mixing density function,

$$\begin{aligned} g(3;3;v) &= \frac{c\pi^{3/2}}{v^{3/2}}\Big[\frac{1}{(a_3^2-a_2^2)^2(a_3^2-a_1^2)^2}\exp\{-\frac{a_3^2}{v}\} \\ &+ \frac{1}{(a_2^2-a_3^2)^2(a_2^2-a_1^2)^2}\exp\{-\frac{a_2^2}{v}\} \\ &+ \frac{1}{(a_1^2-a_2^2)^2(a_1^2-a_3^2)^2}\exp\{-\frac{a_1^2}{v}\}\Big] \\ &+ \frac{2c\pi^{3/2}}{v^{1/2}}\Big[\frac{1}{(a_3^2-a_1^2)^3(a_3^2-a_2^2)^3}\Big(2a_3^2-a_1^2-a_2^2\Big)\exp\{-\frac{a_3^2}{v}\} \\ &+ \frac{1}{(a_2^2-a_1^2)^3(a_2^2-a_3^2)^3}\Big(2a_2^2-a_3^2-a_1^2\Big)\exp\{-\frac{a_2^2}{v}\} \\ &+ \frac{1}{(a_1^2-a_2^2)^3(a_1^2-a_3^2)^3}\Big(2a_1^2-a_2^2-a_3^2\Big)\exp\{-\frac{a_1^2}{v}\}\Big]. \end{aligned} \tag{2}$$

4 The Laplace transform of $g(3;3;v)$

Denote the Laplace transform of the mixing density function $g(3;3;v)$ by $L(3;3;s)$. Then we obtain

$$\begin{aligned} L(3;3;s) &= \int_0^\infty \exp\{-sv\}g(3;3;v)dv \\ &= c\Big[\frac{1}{b_{12}^2b_{13}^2}\Big[\frac{1}{a_1}+\Big\{\frac{1}{b_{12}}+\frac{1}{b_{13}}\Big\}\frac{2}{\sqrt{s}}\Big]\exp\{-2a_1\sqrt{s}\} \\ &+ \frac{1}{b_{21}^2b_{23}^2}\Big[\frac{1}{a_2}+\Big\{\frac{1}{b_{21}}+\frac{1}{b_{23}}\Big\}\frac{2}{\sqrt{s}}\Big]\exp\{-2a_2\sqrt{s}\} \\ &+ \frac{1}{b_{31}^2b_{32}^2}\Big[\frac{1}{a_3}+\Big\{\frac{1}{b_{31}}+\frac{1}{b_{32}}\Big\}\frac{2}{\sqrt{s}}\Big]\exp\{-2a_3\sqrt{s}\}\Big] \end{aligned}$$

where let Re $s \geq 0$ and $a_1 = 1, a_2 = 2, a_3 = 3$ and let $b_{12} = a_1^2 - a_2^2,\ b_{13} = a_1^2 - a_3^2,\ b21 = a_2^2 - a_1^2, b23 = a_2^2 - a_3^2,\ b31 = a_3^2 - a_1^2, b32 = a_3^2 - a_2^2$. It is

$$\begin{aligned} & L(3;3;s)b_{12}^2b_{13}^2/c \\ =\ & \frac{3}{25}\exp(-6\sqrt{s}) + \frac{32}{25}\exp(-4\sqrt{s}) + \exp(-\sqrt{s}) \\ +\ & \frac{117}{500\sqrt{s}}\exp(-6\sqrt{s}) + \frac{256}{375\sqrt{s}}\exp(-4\sqrt{s}) - \frac{11}{12\sqrt{s}}\exp(-\sqrt{s}) \end{aligned}$$

and we take a branch such that $L(3;3;+0) = 1$. We will explain the above result. Denote the Laplace-Stieltjes transform of the mixing density $g(3;3;v)$

by $L(3;3;s)$. We note that under the condition Re $s \geq 0$ the Laplace transform converges and the right hand side is a holomorphic function of the variable s in the right half complex plane. Then we obtain

$$\begin{aligned} L(3;3;s) &= \int_0^\infty \exp\{-sv\} g(3;3;v)dv \\ = & \; c\Big[\frac{1}{b_{12}^2 b_{13}^2}\Big[\frac{1}{a_1} + \Big\{\frac{1}{b_{12}} + \frac{1}{b_{13}}\Big\}\frac{2}{\sqrt{s}}\Big] \exp\{-2a_1\sqrt{s}\} \\ + & \; \frac{1}{b_{21}^2 b_{23}^2}\Big[\frac{1}{a_2} + \Big\{\frac{1}{b_{21}} + \frac{1}{b_{23}}\Big\}\frac{2}{\sqrt{s}}\Big] \exp\{-2a_2\sqrt{s}\} \\ + & \; \frac{1}{b_{31}^2 b_{32}^2}\Big[\frac{1}{a_3} + \Big\{\frac{1}{b_{31}} + \frac{1}{b_{32}}\Big\}\frac{2}{\sqrt{s}}\Big] \exp\{-2a_3\sqrt{s}\}\Big] \end{aligned}$$

By analytic continuation we obtain the Laplace transform $L(3;3;s)$ in the complex plane except the origin and we take the branch such that $L(3;3;+0) = 1$. We obtain

$$\begin{aligned} & L(3;3;s) b_{12}^2 b_{13}^2 / c \\ = & \; \frac{3}{25}\exp(-6\sqrt{s}) + \frac{32}{25}\exp(-4\sqrt{s}) + \exp(-2\sqrt{s}) \\ + & \; \frac{117}{500\sqrt{s}}\exp(-6\sqrt{s}) + \frac{256}{375\sqrt{s}}\exp(-4\sqrt{s}) - \frac{11}{12\sqrt{s}}\exp(-2\sqrt{s}) \end{aligned}$$

concretely and write it

$$\begin{aligned} L(3;z) &= \frac{3}{25}\exp(-6z) + \frac{32}{25}\exp(-4z) + \exp(-2z) \\ + & \; \frac{117}{500z}\exp(-6z) + \frac{256}{375z}\exp(-4z) - \frac{11}{12z}\exp(-2z) \end{aligned} \qquad (3)$$

by replacing $\sqrt{s}$ by z. We note that

$$\frac{117}{500} + \frac{256}{375} - \frac{11}{12} = 0$$

and

$$\frac{3}{25} + \frac{32}{25} + 1 + \frac{117}{500}(-6) + \frac{256}{375}(-4) - \frac{11}{12}(-2) = \frac{37}{375}.$$

From these facts it is seen that $L(3;z)$ is bounded and does not vanish in the neighborhood of origin.

5 The inverse Laplace transform and spectral function

Next we will talk about the inverse Laplace transform to obtain $k(t)$. We make use of the polar coordinate

$$s = \rho e^{i\theta}, \ \ (-\pi < \theta < \pi, 0 < \rho).$$

Then

$$\sqrt{s} = \sqrt{\rho}e^{i\theta/2} = \sqrt{\rho}(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2})$$

and $0 < \sqrt{\rho}\cos(\theta/2)$. Let us calculate the spectral function $k(t)$ by the formula

$$k(t) = \lim_{R\to\infty}\frac{1}{2\pi i}\int_{\xi-iR_1}^{\xi+iR_1} e^{ts}(-1)\frac{L'(3;3;s)}{L(3;3;s)}ds,$$
$$(0<\xi,\ \epsilon;\ 0<t;\quad R_1 = R\cos\epsilon),$$

on the figure after the References. Since $\sqrt{s} = z$ we see that

$$\frac{dL(3;3;s)}{ds}\frac{1}{L(3;3;s)} = \frac{dL(3;z)}{dz}\frac{1}{2zL(3;z)}$$

and

$$k(t) = \lim_{R\to\infty}\frac{1}{2\pi i}\int_{\xi-iR_1}^{\xi+iR_1} e^{ts}(-1)\frac{L'(3;z)}{L(3;z)}\frac{ds}{2\sqrt{s}}, \tag{4}$$

From computations we can see that the function $L(3;z)$ does not vanish in the bounded region near at origin in the right half complex plane. And we see that as $|z|\to\infty$

$$L(3;z) \approx \frac{3}{25}\exp(-6z) + \frac{32}{25}\exp(-4z) + \exp(-2z)$$

and if we put

$$Z := \exp(-2z)$$

we obtain at most two roots from the equation

$$\frac{3}{25}Z^2 + \frac{32}{25}Z + 1 = 0$$

and if $|z| \geq R$ and R is sufficiently large the function $L(3;z)$ does not have zero points in the right half complex plane. From the Cauchy theorem we obtain

$$\int_\gamma e^{ts}(-1)\frac{L'(3;z)}{L(3;z)}\frac{ds}{2\sqrt{s}} = 0$$

where γ is a curve of the figure and hence we obtain the following equality

$$\begin{aligned}
&\int_{A\to B} e^{ts}(-1)\frac{L'(3;z)}{L(3;z)}\frac{ds}{2\sqrt{s}}\\
= &\ -\int_{B\frown C} e^{ts}(-1)\frac{L'(3;z)}{L(3;z)}\frac{ds}{2\sqrt{s}} - \int_{C\frown D} e^{ts}(-1)\frac{L'(3;z)}{L(3;z)}\frac{ds}{2\sqrt{s}}\\
- &\ \int_{D\to G} e^{ts}(-1)\frac{L'(3;z)}{L(3;z)}\frac{ds}{2\sqrt{s}} - \int_{G\frown H} e^{ts}(-1)\frac{L'(3;z)}{L(3;z)}\frac{ds}{2\sqrt{s}}
\end{aligned}$$

$$- \int_{H\to E} e^{ts}(-1)\frac{L'(3;z)}{L(3;z)}\frac{ds}{2\sqrt{s}} - \int_{E\frown F} e^{ts}(-1)\frac{L'(3;z)}{L(3;z)}\frac{ds}{2\sqrt{s}}$$
$$- \int_{F\frown A} e^{ts}(-1)\frac{L'(3;z)}{L(3;z)}\frac{ds}{2\sqrt{s}}.$$

It holds that (S1)

$$\int_{B\frown C} e^{ts}(-1)\frac{L'(3;z)}{L(3;z)}\frac{ds}{2\sqrt{s}} \to 0,$$

(S2)

$$\int_{F\frown A} e^{ts}(-1)\frac{L'(3;z)}{L(3;z)}\frac{ds}{2\sqrt{s}} \to 0,$$

(S3)

$$\int_{C\frown D} e^{ts}(-1)\frac{L'(3;z)}{L(3;z)}\frac{ds}{2\sqrt{s}} \to 0,$$

(S4)

$$\int_{E\frown F} e^{ts}(-1)\frac{L'(3;z)}{L(3;z)}\frac{ds}{2\sqrt{s}} \to 0$$

as $R \to \infty$ and (S5)

$$\int_{G\frown H} e^{ts}(-1)\frac{L'(3;z)}{L(3;z)}\frac{ds}{2\sqrt{s}} \to 0$$

as $r \to 0$. Therefore we obtain

$$k(t) = \lim_{R\to\infty} \frac{1}{2\pi i}\int_{\xi-iR_1}^{\xi+iR_1} e^{ts}(-1)\frac{L'(3;z)}{L(3;z)}\frac{ds}{2\sqrt{s}}$$
$$= \lim_{R\to\infty,\, r\to 0}\{\frac{1}{2\pi i}\int_{D\to G} e^{ts}\frac{L'(3;z)}{L(3;z)}\frac{ds}{2\sqrt{s}}$$
$$+\frac{1}{2\pi i}\int_{H\to E} e^{ts}\frac{L'(3;z)}{L(3;z)}\frac{ds}{2\sqrt{s}}\}.$$

and at last we obtain

$$k(t)$$
$$= \lim_{R\to\infty,\, r\to 0}\{\frac{1}{2\pi i}\int_R^r e^{t\rho e^{i\pi}}\frac{L'(3;\sqrt{\rho}e^{i\pi/2})}{2\sqrt{\rho}e^{i\pi/2}L(3;\sqrt{\rho}e^{i\pi/2})}e^{i\pi}d\rho$$
$$+\frac{1}{2\pi i}\int_r^R e^{t\rho e^{-i\pi}}\frac{L'(3;\sqrt{\rho}e^{-i\pi/2})}{2\sqrt{\rho}e^{-i\pi/2}L(3;\sqrt{\rho}e^{-i\pi/2})}e^{-i\pi}d\rho\}$$
$$= \lim_{R\to\infty,\, r\to 0}\{\frac{1}{2\pi i}\int_r^R e^{-t\rho}\frac{L'(3;\sqrt{\rho}i)}{2\sqrt{\rho}i\, L(3;\sqrt{\rho}i)}d\rho$$

$$+\frac{1}{2\pi i}\int_r^R e^{-t\rho}\frac{L'(3;-i\sqrt{\rho})}{2\sqrt{\rho}(-i)\ L(3;-\sqrt{\rho}i)}(-1)d\rho\}$$
$$= \lim_{R\to\infty,\ r\to 0}\{-\frac{1}{2\pi}\int_r^R e^{-t\rho}\frac{L'(3;\sqrt{\rho}i)}{2\sqrt{\rho}\ L(3;\sqrt{\rho}i)}d\rho$$
$$-\frac{1}{2\pi}\int_r^R e^{-t\rho}\frac{L'(3;-\sqrt{\rho}i)}{2\sqrt{\rho}\ L(3;-\sqrt{\rho}i)}d\rho\}$$
$$= \lim_{R\to\infty,\ r\to 0}\{-\frac{1}{2\pi}\int_r^R e^{-t\rho}(\frac{L'(3;\sqrt{\rho}i)}{L(3;\sqrt{\rho}i)}+\frac{L'(3;-i\sqrt{\rho})}{L(3;-\sqrt{\rho}i)})\frac{d\rho}{2\sqrt{\rho}}\}$$

From the above we see that

$$k(t) = \frac{1}{2\pi}\int_0^\infty e^{-t\rho}$$
$$(-1)\frac{L'(3;\sqrt{\rho}i)L(3;-\sqrt{\rho}i)+L'(3;-\sqrt{\rho}i)L(3;\sqrt{\rho}i)}{L(3;\sqrt{\rho}i)L(3;-\sqrt{\rho}i)}\frac{d\rho}{2\sqrt{\rho}} \quad (5)$$

It is necessary to show that

$$N = (-1)\{L'(3;\sqrt{\rho}i)L(3;-\sqrt{\rho}i)+L'(3;-\sqrt{\rho}i)L(3;\sqrt{\rho}i)\}$$

is non-negative and by computations we obtain

$$N = (-1)\{L'(3;\sqrt{\rho}i)L(3;-\sqrt{\rho}i)+L'(3;-\sqrt{\rho}i)L(3;\sqrt{\rho}i)\}$$
$$= \frac{4}{125y^2}\Big(243+540y^2+64(-2+9y^2)\cos(2y)$$
$$+ 5(-23+12y^2)\cos(4y)-832\sin(2y)-172\sin(4y)\Big). \quad (6)$$

Here y equals $\sqrt{\rho}$. It is seen that as $y\to\infty$,

$$N/2 \approx \frac{216}{25}+\frac{1152}{125}\cos(2y)+\frac{24}{25}\cos(4y)$$
$$= \frac{48}{125}+\frac{1344}{125}(\cos(y))^2+\frac{192}{25}(\cos(y))^4. \quad (7)$$

By making use of the computer we can see that N is positive in the bounded interval like the interval $(0,l)$ and therefore the author concludes that the mixing density function $g(3;3;v)$ is infinitely divisible and consequently that the variance mixture density of normed product of the triple Cauchy densities (1) is infinitely divisible.

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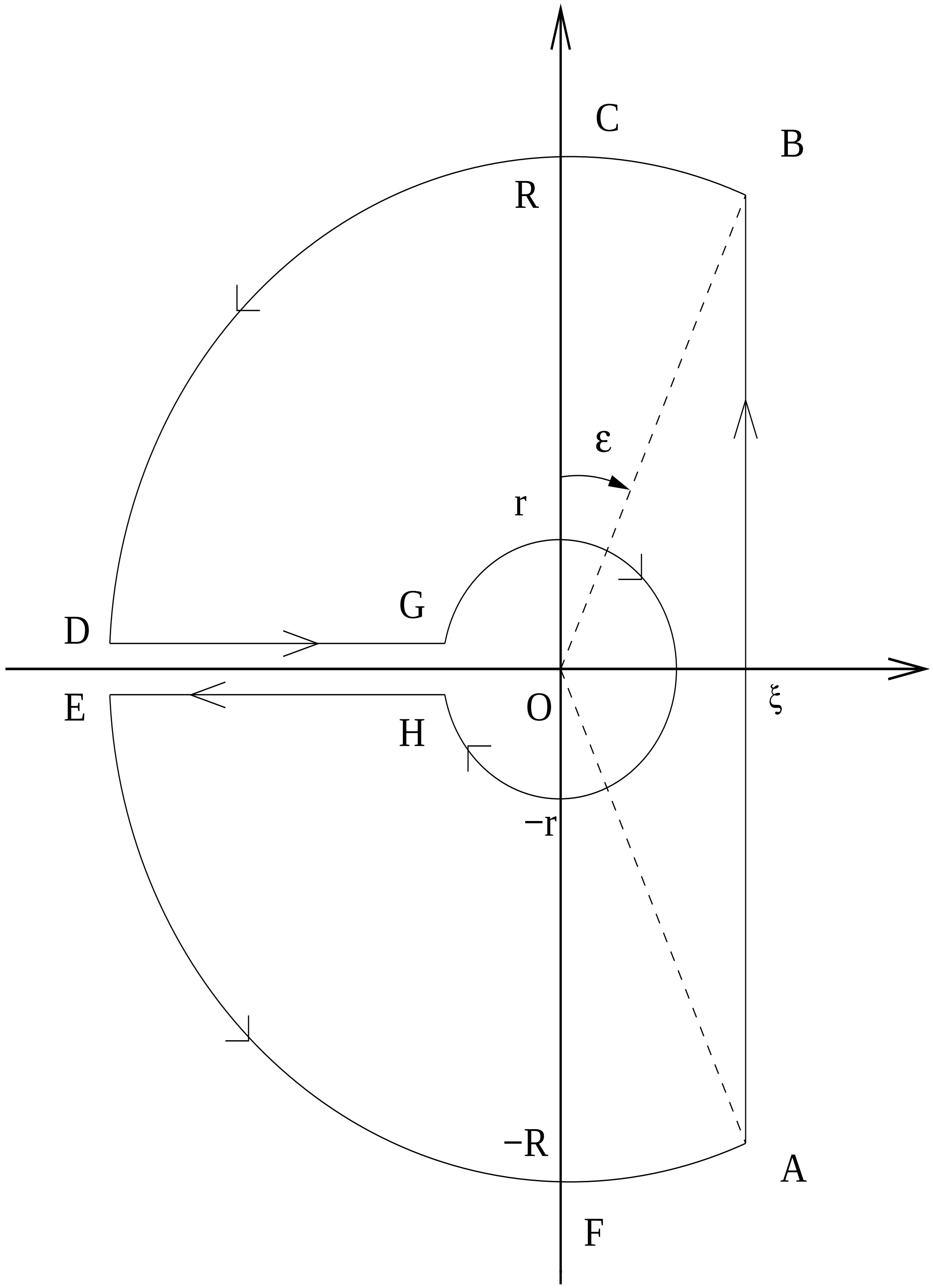
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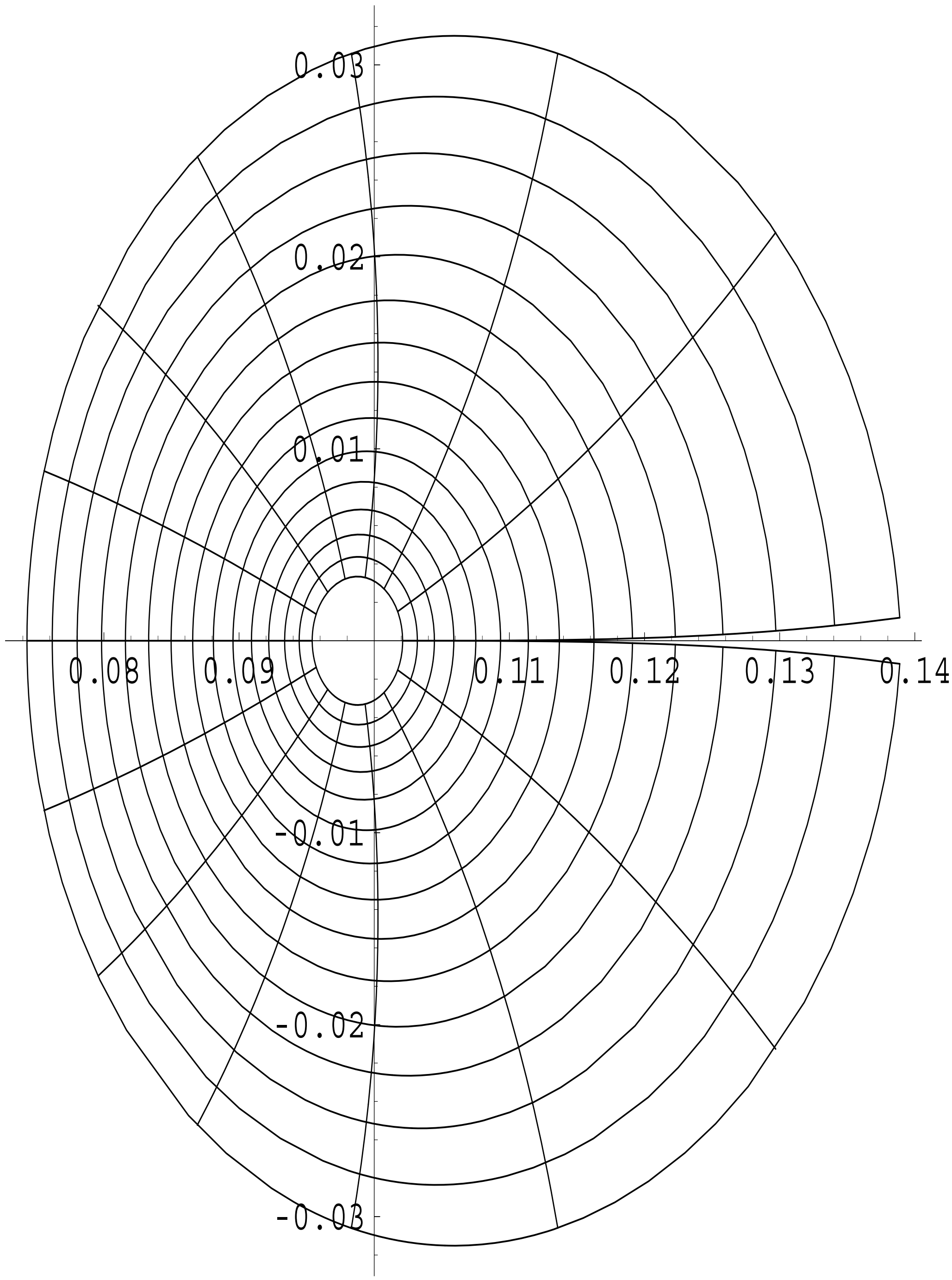
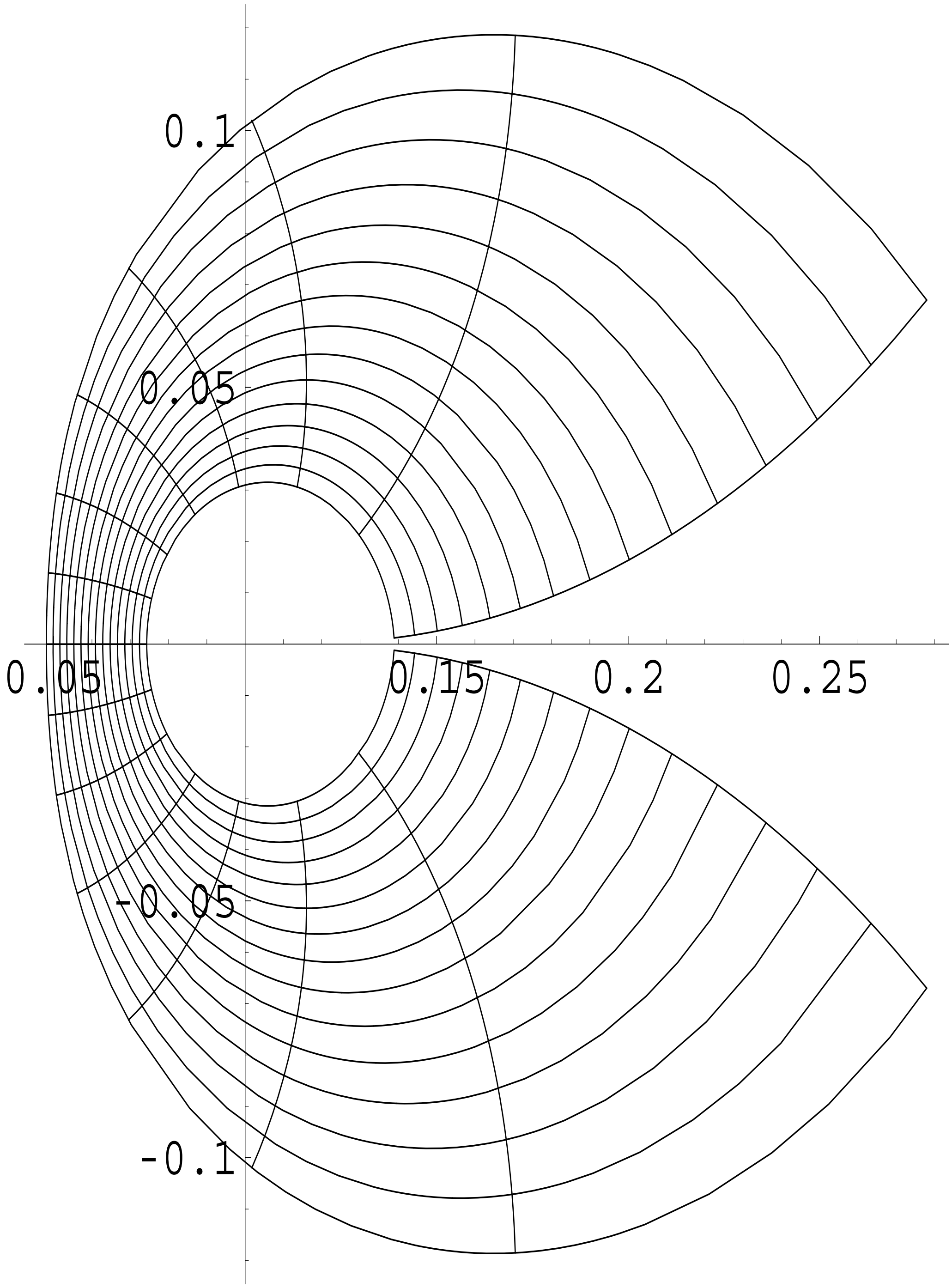
PolarMap of $L(3;z)$

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-0.05
-0.1
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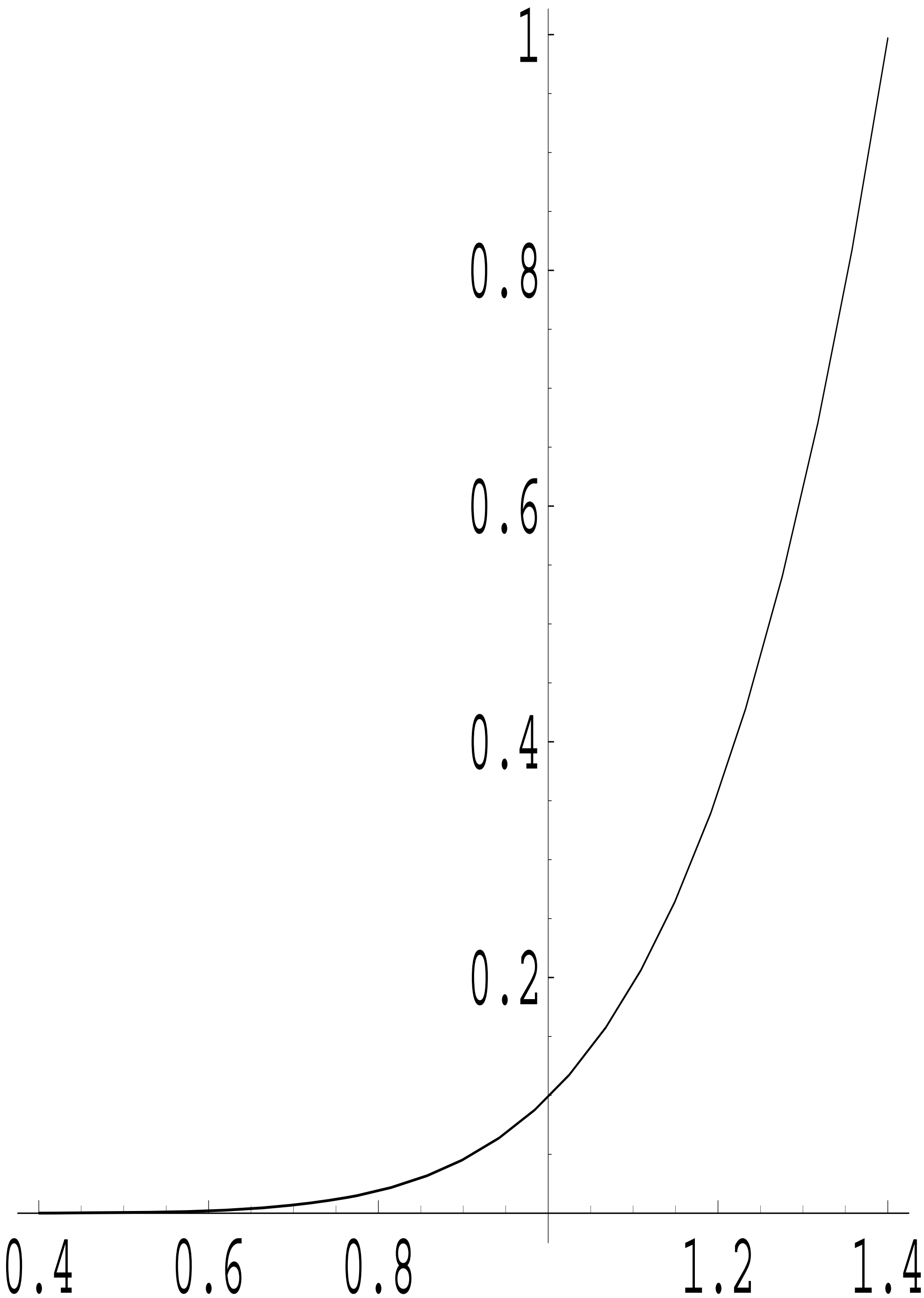
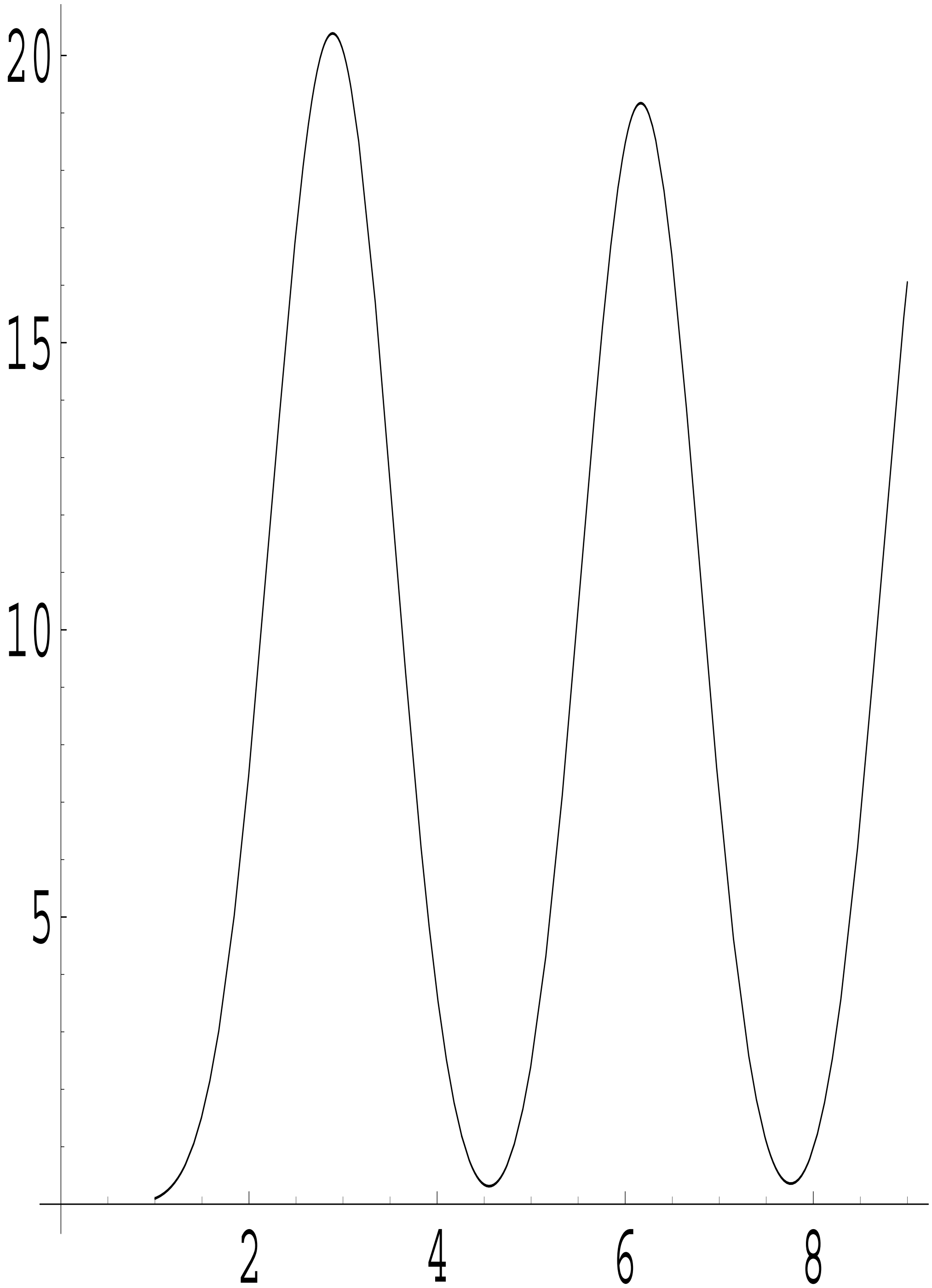
0.25

Graphics of $N(y)/2$





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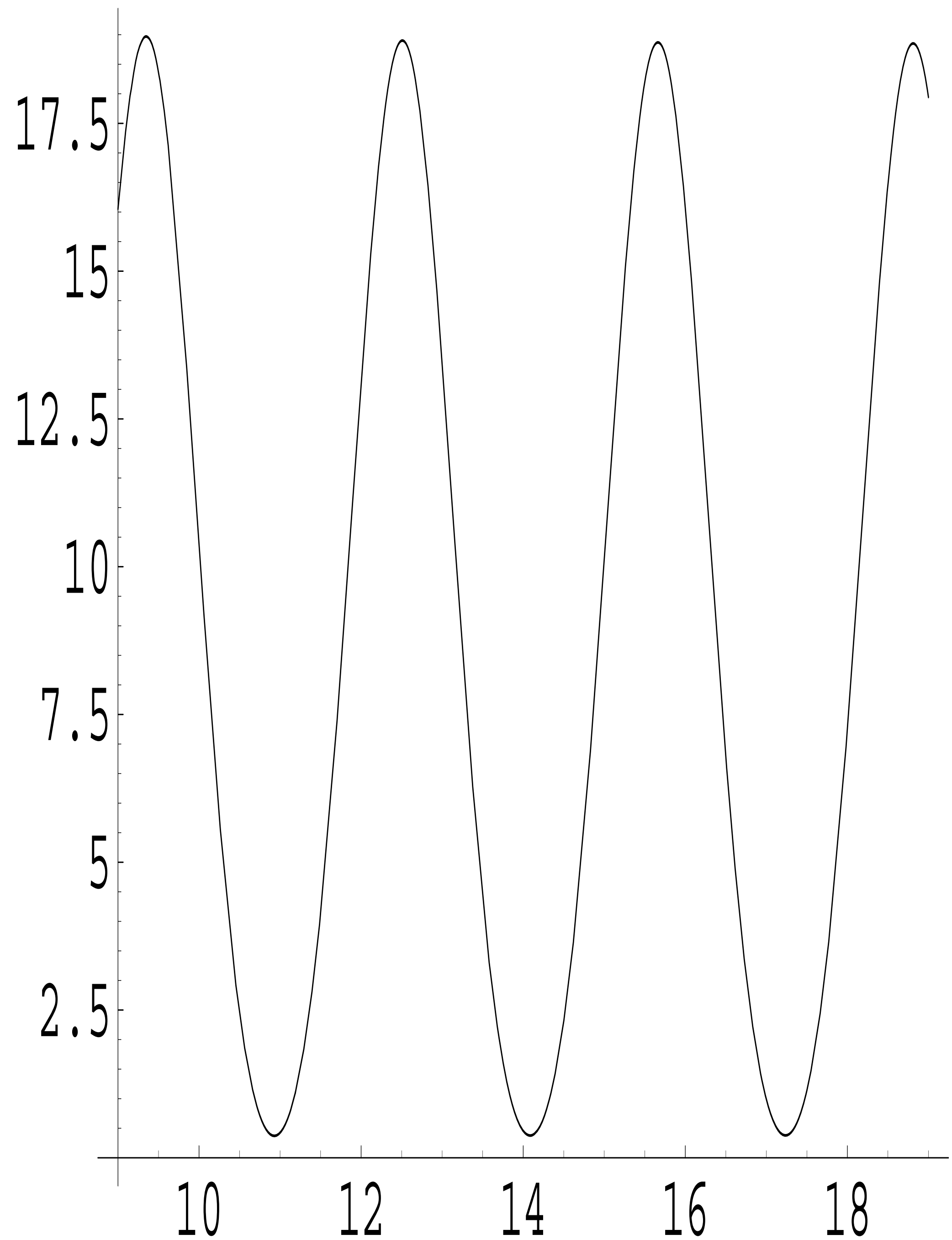
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