



2015 HAWAII UNIVERSITY INTERNATIONAL CONFERENCES

S.T.E.A.M. & EDUCATION JUNE 13 - 15, 2015

ALA MOANA HOTEL, HONOLULU, HAWAII

S.T.E.A.M & EDUCATION PUBLICATION:

ISSN 2333-4916 (CD-ROM)

ISSN 2333-4908 (ONLINE)

THE DOUBLE ANGLE FORMULA OF THE MULATU NUMBERS

MULATU, LEMMA & ET AL

SAVANNAH STATE UNIVERSITY, GEORGIA

DEPARTMENT OF MATHEMATICS

The Double Angle Formula of the Mulatu Numbers

Mulatu Lemma, Jonathan Lambright and Ermi Asfaw

College of Science and Technology

Savannah State University

Savannah, GA 31404, U.S.A.

Hawaii University International Conference

Abstract: The Mulatu numbers were introduced in [1]. The numbers are sequences of numbers of the form: **4,1, 5,6,11,17,28,45...** The numbers have wonderful and amazing properties and patterns.

In mathematical terms, the sequence of the Mulatu numbers is defined by the following recurrence relation:

$$M_n := \begin{cases} 4 & \text{if } n = 0; \\ 1 & \text{if } n = 1; \\ M_{n-1} + M_{n-2} & \text{if } n > 1. \end{cases}$$

The double Angel Formulas for Fibonacci and Lucas numbers are given by the following formulas respectively.

$$(1) F_{2n} = F_n L_n \text{ and } (2) L_{2n} = \frac{5F_n^2 + L_n^2}{2}$$

Since both the Fibonacci and Lucas numbers have double angle Formulas, It is natural to ask if such formula exist for Mulatu Numbers. The answer is affirmative and produces the following papers.

2000 Mathematical Subject Classification: 11

Key Words: Mulatu numbers, Mulatu sequences, Fibonacci numbers, Lucas numbers, Fibonacci sequences, and Lucas sequences.

- 1. Introduction and Background.** As given in [1], the Mulatu numbers are a sequence of numbers recently introduced by Mulatu Lemma, Professor of Mathematics at Savannah State University, Savannah, Georgia, USA. The Mulatu sequence has wealthy mathematical properties and patterns like the two celebrity sequences of Fibonacci and Lucas.

In this paper, more interesting relationships of the Mulatu numbers to the Fibonacci and Lucas numbers will be presented.

Here are the First 21 Mulatu, Fibonacci, and Lucas numbers for quick reference.

**Mulatu(M_n), Fibonacci(F_n) and Lucas(L_n) Numbers
(Tables 1 & 2)**

Table 1

| | | | | | | | | | | | | |
|-----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|-----------|-----------|
| n: | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| M_n : | 4 | 1 | 5 | 6 | 11 | 17 | 28 | 45 | 73 | 118 | 191 | 309 |
| F_n : | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 |
| L_n : | 2 | 1 | 3 | 4 | 7 | 11 | 18 | 29 | 47 | 76 | 123 | 199 |

Table 2

| | | | | | | | | | |
|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| n: | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| M_n | 500 | 809 | 1309 | 2118 | 3427 | 5545 | 8972 | 14517 | 23489 |
| F_n : | 144 | 233 | 377 | 610 | 987 | 1597 | 2584 | 4181 | 6765 |
| L_n : | 322 | 521 | 843 | 1364 | 2207 | 3571 | 5778 | 9349 | 15127 |

Remark 1 : Throughout this paper M, F, and L stand for Mulatu numbers, Fibonacci numbers, and Lucas number respectively.

The following well-known identities of Mulatu numbers, Fibonacci numbers, and Lucas numbers are required in this paper and hereby listed for quick reference.

$$(1) L_n = F_{n-1} + F_{n+1}$$

$$(2) F_{n+1} = F_n + F_{n-1}$$

$$(3) F_{2n} = F_n L_n$$

$$(4) L_{2n} = F_n + 2F_{n-1}$$

$$(5) F_n = \frac{L_{n+1} + L_{n-1}}{5}$$

$$(6) L_{n+1} = L_n + L_{n-1}$$

$$(7) F_{n+k} = F_{n-1}F_k + F_nF_{k+1}$$

$$(8) 5F_n^2 - L_n^2 = 4(-1)^{n+1}$$

$$(9) L_{n+m} = \frac{5F_nF_m + L_nL_m}{2}$$

$$(10) M_{n+k} = F_{n-1}M_k + M_nF_{k+1}$$

The Main Results.

We will state the following theorem proved in [1] as proposition 1 and use it.

Proposition 1. $M_n = F_{n-3} + F_{n-1} + F_{n+2}$

Theorem 1: The following are equivalent.

- (1) M_n
- (2) $F_{n-3} + F_{n-1} + F_{n+2}$
- (3) $L_n + 2F_{n-1}$
- (4) $F_n + 4F_{n-1}$
- (5) $4F_{n+1} - 3F_n$

Proof: We will show that (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (1)

(i) (1) \Rightarrow (2) follows by Proposition 1.

(ii) (2) \Rightarrow (3) follows as shown:

$$\begin{aligned} F_{n-3} + F_{n-1} + F_{n+2} &= F_{n-3} + F_{n-1} + F_{n+1} + F_n \\ &= F_{n-3} + F_{n-1} + F_{n+1} + F_{n-1} + F_{n-2} \\ &= F_{n-1} - F_{n-2} + F_{n-1} + F_{n+1} + F_{n-1} + F_{n-2} \\ &= 2F_{n-1} + L_n \end{aligned}$$

(iii) (3) \Rightarrow (4) follows as shown:

$$\begin{aligned} L_n + 2F_{n-1} &= F_{n+1} + F_{n-1} + 2F_{n-1} \\ &= F_n + F_{n-1} + F_{n-1} + 2F_{n-1} \\ &= F_n + 4F_{n-1} \end{aligned}$$

(iv) (4) \Rightarrow (5) follows as shown:

$$F_n + 4F_{n-1} = F_n + 4(F_{n+1} - F_n)$$

$$= 4F_{n+1} - 3F_n$$

(v) (5) \Rightarrow (1) follows as shown:

$$\begin{aligned} 4F_{n+1} - 3F_n &= 4F_{n+1} - 3(F_{n+1} - F_{n-1}) = F_{n+1} + 3F_{n-1} = F_{n+1} + F_{n-1} + F_{n-1} + F_{n-1} \\ &= F_{n+1} + (F_n - F_{n-2}) + F_{n-1} + F_{n-3} + F_{n-2} = F_{n+1} + F_n - F_{n-2} + F_{n-1} + F_{n-3} + F_{n-2} \\ &= F_{n+2} + F_{n-1} + F_{n-3} = M_n \text{ by Proposition 1 and hence) (5) } \Rightarrow \text{(1). Thus} \end{aligned}$$

the theorem is proved.

Theorem 2: $L_n^2 = F_{n+1}(M_n + F_n) - F_{2n}$

$$\begin{aligned} \text{Proof: } L_n^2 &= (F_n + 2F_{n-1})^2 = F_n^2 + 4F_n F_{n-1} + 4F_{n-1}^2 \\ &= -F_n(F_n + 2F_{n-1}) + (F_n + F_{n-1})(F_n + 4F_{n-1}) + F_n^2 + F_n F_{n-1} \\ &= -F_n L_n + F_{n+1} M_n + F_n(F_n + F_{n-1}) \\ &= -F_n L_n + F_{n+1} M_n + F_{n+1} F_n \\ &= -F_n L_n + F_{n+1}(M_n + F_n) \\ &= F_{n+1}(M_n + F_n) - F_n L_n \\ &= F_{n+1}(M_n + F_n) - F_{2n} \end{aligned}$$

Hence the theorem is proved.

Theorem 3. $M^2 = F_{2n} + 6F_{n-1}F_n + 16F_{n-1}^2$

$$\begin{aligned} \text{Proof: } M^2 &= MM = (L_n + 2F_{n-1})(F_n + 4F_{n-1}) \\ &= L_n F_n + 4F_{n-1}(F_n + 2F_{n-1}) + 2F_{n-1}F_n + 8F_{n-1}^2 \\ &= F_{2n} + 6F_{n-1}F_n + 8F_{n-1}^2 + 8F_{n-1}^2 \\ &= F_{2n} + 6F_{n-1}F_n + 16F_{n-1}^2 \end{aligned}$$

Hence, the theorem is proved.

Lemma 1. $F_{2n-1} = F_n^2 + F_{n-1}^2$

Proof: Applying (7) above, we have $F_{2n-1} = F_{n+n-1} = F_{n-1}^2 + F_n^2$

Lemma 2. $M_{n+1} = F_{n-1} + 5F_n$

Proof: using (10) above, we have

$$M_{n+1} = F_{n+1}M_1 + 5F_nM_0 = F_{n-1} + 5F_n.$$

Theorem 4. The following are equivalent.

1. M_{2n}
2. $F_{2n} + 4F_{2n-1}$
3. $4F_{2n+1} - 3F_{2n}$
4. $L_{2n} + 2F_{2n-1}$
5. $\frac{9F_n^2 + L_n^2 + 4F_{n-1}^2}{2}$
6. $M_n L_n + 5F_n^2 - L_n^2$

Proof (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) follows by Theorem 1. We will suffice to show that (4) \Rightarrow (5) \Rightarrow (6) \Rightarrow (1)

(i) (4) \Rightarrow (5). Note that Using (9) above and lemma 1, we have $L_{2n} + 2F_{2n-1} =$

$$\frac{5F_n^2 + L_n^2}{2} + 2F_n^2 + 2F_{n-1}^2 = \frac{9F_n^2 + L_n^2 + 4F_{n-1}^2}{2}.$$

(ii) (5) \Rightarrow (6). We show this using Theorem 3 and Lemma 1.

Note that

$$\begin{aligned} \frac{9F_n^2 + L_n^2 + 4F_{n-1}^2}{2} &= \frac{5F_n^2 + L_n^2}{2} + 2F_n^2 + 2F_{n-1}^2 \\ &= L_{2n} + 2F_{2n-1} = F_{2n} + 2F_{2n-1} + 2F_{2n-1} = F_n L_n + 4F_{2n-1} \\ &= F_n (F_n + 2F_{n-1}) + 4F_{n-1}^2 + 4F_n^2 = 5F_n^2 + 4F_{n-1}^2 + 2F_{n-1} F_n = \\ &= (F_n^2 + 8F_{n-1}^2 + 6F_{n-1} F_n) + 5F_n^2 - (F_n^2 + 4F_{n-1} F_n + 4F_{n-1}^2) \\ &= (F_n + 4F_{n-1})(F_n + 2F_{n-1}) + 5F_n^2 - (F_n + 2F_{n-1})^2 \\ &= M_n L_n + 5F_n^2 - L_n^2 \end{aligned}$$

(iii) (6) \Rightarrow (1). We show this using Theorem 3 and Lemma 2.

Note that

$$\begin{aligned} M_n L_n + 5F_n^2 - L_n^2 &= M_n L_n - L_n^2 + 5F_n^2 \\ &= M_n L_n - (F_n + 2F_{n-1})^2 + 5F_n^2 \\ &= M_n L_n - (F_n^2 + 4F_{n-1} F_n + 4F_{n-1}^2) + 5F_n^2 \\ &= M_n L_n - (F_n + F_{n-1})(F_n + 4F_{n-1}) + F_n F_{n-1} + 5F_n^2 \\ &= M_n L_n - F_{n+1} M_n + F_n F_{n-1} + 5F_n^2 \\ &= M_n (L_n - F_{n+1}) + F_n (F_{n+1} + 5F_n) \end{aligned}$$

$$\begin{aligned}
&= M_n F_{n-1} + F_n M_{n+1} \\
&= M_{n+n} = M_{2n}.
\end{aligned}$$

Hence, (6) \Rightarrow (1) and the theorem is proved.

Corollary 1: $M_{2n} = M_n L_n + 4(-1)^{n+1}$

Proof: The corollary easily follows by Theorem 4, using (8) above.

References:

1. Mulatu Lemma, The Mulatu Numbers, *Advances and Applications in Mathematical Sciences*, Volume 10, issue 4, August 2011, page 431-440.
2. Burton, D. M., *Elementary number theory*. New York City, New York: McGraw-Hill. 1998.