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Abstract

A *Continued Fraction* is of the form, $a + \cfrac{b}{c + \cfrac{d}{e + \cfrac{f}{\ddots}}}$ where a, b ,c ,d, e, f, $\ldots, \in \mathbb{Z}$. In this paper, we derive formulas for the n'th convergent of the C.F.'s $p + \cfrac{1}{p + \cfrac{1}{p + \cfrac{1}{\ddots}}}$ and $1 + \cfrac{1}{p + \cfrac{1}{1 + \cfrac{1}{\ddots}}}$ where $p \in \mathbb{Z}$. The associated number sequences and electrical networks are indicated.

1 Preliminaries

A *Continued Fraction (C.F.)* is of the form, $a + \cfrac{b}{c + \cfrac{d}{e + \cfrac{f}{\ddots}}}$ where a, b ,c ,d, e, f, $\ldots, \in \mathbb{Z}$. While there are many variants to the form of these C.F.'s, we are interested in those whose terms a, b, c, d, e, f, $\ldots$, $\in \mathbb{N}$. The first, second and third convergent of the above C.F.

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are respectively, $\frac{a}{1}$, $a+\frac{1}{b}=\frac{ab+1}{b}$, $a+\frac{1}{b+\frac{1}{c}}=a+\frac{c}{bc+1}=\frac{a(bc+1)+c}{bc+1}$. The numerator and the denominator of the n'th convergent of the above C.F. will be denoted by N_n and D_n respectively. On careful scrutiny of the convergents, one can write $N_n=\alpha N_{n-1}+\beta N_{n-2}$ and $D_n=\gamma D_{n-1}+\delta D_{n-2}$, ([6], [3]), where, α, β, γ, and δ are the appropriate terms. Let F denote the C.F. $p+\cfrac{1}{p+\cfrac{1}{p+\cfrac{1}{\ddots}}}$ where $p\in\mathbb{Z}$. and G denote the C.F. $1+\cfrac{1}{p+\cfrac{1}{1+\cfrac{1}{1+\ddots}}}$, where $p\in\mathbb{Z}$.

We now divide the rest of the paper into sections containing results for F, G and their application to physics, respectively. It may be noted that an application to a variant of F has been discussed in [1] and [2]. The following results are for C.F.'s F and G which are different from those studied earlier.

2 Result for F

In this section we develop the necessary tools to derive the numerator and denominator of the n'th convergent of F and hence use them to give an explicit expression for the n'th convergent. We also verify certain well known facts about F for specific values of p using the main result. We now need the following Lemmas.

Lemma 1 *[6] For the C.F. F, the numerator and the denominator of the n'th convergent are given by $N_n=pN_{n-1}+N_{n-2}$ with $N_1=p$, $N_2=p^2+1$ and $D_n=pD_{n-1}+D_{n-2}$, with $D_1=1$ and $D_2=p$, respectively.*

Lemma 2 *[4] For Fibonacci type numbers, the n'th term F_n satisfies*

$$F_n=c(\frac{\phi+\sqrt{4+\phi^2}}{2})^n+d(\frac{\phi-\sqrt{4+\phi^2}}{2})^n$$

with a given F_1 and F_2 where, $c,d,\phi\in\mathbb{R}$ are appropriate constants.

Lemma 3 *Let $A_n=\frac{N_n}{D_n}$, where,*

$$D_n=\frac{1}{\sqrt{p^2+4}}(\frac{p+\sqrt{p^2+4}}{2})^n-\frac{1}{\sqrt{p^2+4}}(\frac{p-\sqrt{p^2+4}}{2})^n \tag{1}$$

and

$$N_n=0.5(1+\frac{p}{\sqrt{p^2+4}})(\frac{p+\sqrt{p^2+4}}{2})^n+0.5(1-\frac{p}{\sqrt{p^2+4}})(\frac{p-\sqrt{p^2+4}}{2})^n. \tag{2}$$

Then

$$\lim_{n\to\infty} A_n = \alpha'(\frac{p+\sqrt{p^2+4}}{2}) + \beta'(\frac{p-\sqrt{p^2+4}}{2}),$$

where $\alpha' = 1$ *and* $\beta' = 0$.

PROOF.Let A_n, N_n, D_n be defined as above. Then, it is easy to verify the result for A_n using elementary calculus. □

Theorem 4 *For the C.F. F, the numerator and denominator of the n'th convergent are given respectively by (2) and (1).*

PROOF.By Lemma 1, the C.F. F has the numerator of the n'th convergent, N_n satisfying the general recursive equation

P(n)=pP(n-1)+P(n-2) (3)

where $P(1) = p$ and $P(2) = p^2 + 1$. Similarly, one can see that the denominator, D_n of the n'th convergent satisfies the same recursive equation (3) with the initial conditions $P(1) = 1$ and $P(2) = p$. (For ease of computation, we replace N_n and D_n by $P(n)$ temporarily.)
From Lemmas 3 and 2, one can replace the general expression for N_n and D_n by

$$c(\frac{p+\sqrt{p^2+4}}{2})^n + d(\frac{p-\sqrt{p^2+4}}{2})^n \tag{4}$$

for appropriate constants c and d.
Case I : $P(1) = 1$, $P(2) = p$
Substituting $n = 1$ and $n = 2$ in (4) gives

$$1 = P(1) = c(\frac{p+\sqrt{p^2+4}}{2}) + d(\frac{p-\sqrt{p^2+4}}{2})$$

and

$$p = P(2) = c(\frac{p+\sqrt{p^2+4}}{2})^2 + d(\frac{p-\sqrt{p^2+4}}{2})^2.$$

Solving for c and d, we get $c = \frac{1}{\sqrt{p^2+4}}$ and $d = -\frac{1}{\sqrt{p^2+4}}$.
Case II : $P(1) = p$ and $P(2) = p^2 + 1$
Similar to Case I, substituting $n = 1$ and $n = 2$ in (4), we get

$$p = P(1) = c(\frac{p+\sqrt{p^2+4}}{2}) + d(\frac{p-\sqrt{p^2+4}}{2})$$

and

$$p^2+1 = P(2) = c(\frac{p+\sqrt{p^2+4}}{2})^2 + d(\frac{p-\sqrt{p^2+4}}{2})^2.$$

We now see that $c = 0.5(1+\frac{p}{\sqrt{p^2+4}})$ and $d = 0.5(1-\frac{p}{\sqrt{p^2+4}})$. □

It can be observed that substituting $p = 1$ in F gives the Fibonacci numbers for $P(n)$. As a result, Theorem 4 gives the limiting value of the ratios of successive terms of the numerator and the denominator respectively of the n'th convergent to be the golden mean.
Similarly, substituting $p = 2$ in F gives the Pell numbers for P(n). As a result, Theorem 4 gives the limiting value of the ratios of successive terms of the numerator and the denominator respectively of the n'th convergent to be the silver mean.
The Binet's formula for Fibonacci Numbers is given by the expression $\frac{\phi^n - \overline{\phi}^{-n}}{\sqrt{5}}$ where ϕ is the golden ratio and $\overline{\phi}$, its conjucgate. Analogous to Binet's form, we get the following result for the numerator and denominator of F.

Corollary 5 *Letting $\phi = \frac{p+\sqrt{p^2+4}}{2}$ and $\overline{\phi}$, its conjugate, the numerator and denominator of F from 4 are $\frac{\phi^n - \overline{\phi}^n}{\phi - \overline{\phi}}$ and $\frac{\phi^{n+1} - \overline{\phi}^{n+1}}{\phi - \overline{\phi}}$ respectively.*

Proof. Clear. □

3 Result for G

In order to study certain numbers related to the Fibonacci numbers like Lucas numbers etc, consider a variant to the C.F. F, namely, the C.F. G defined earlier.

Lemma 6 *For the C.F. G, the numerator and the denominator of the n'th convergent are given by $N_n = N_{n-1} + N_{n-2}$ with $N_1 = 1$, $N_2 = p+1$ and $D_n = D_{n-1} + D_{n-2}$, with $D_1 = 1$ and $D_2 = p$, respectively.*

Any C.F. of the form of G has the numerator or denominator of the n'th convergent looking like

$$c(\frac{1+\sqrt{5}}{2})^n + d(\frac{1-\sqrt{5}}{2})^n, \tag{5}$$

WLOG, assume that N_n and D_n to be of the form (5) for appropriate constants c and d.

Theorem 7 *For the C.F. G, the numerator and denominator of the n'th convergent are respectively*

$$N_n = \frac{p(\sqrt{5}-1)+2}{2\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n + \frac{p(\sqrt{5}+1)-2}{2\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$$

and

$$D_n = \frac{p(\sqrt{5}-1)+3-\sqrt{5}}{2\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n + \frac{p(\sqrt{5}+1)-3-\sqrt{5}}{2\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n.$$

Proof.Let G be the C.F. defined as above. By Lemma 6, the numerator and denominator satisfy the general recursive equation

$$P(n) = P(n-1) + P(n-2) \tag{6}$$

with the initial conditions $P(1) = 1$, $P(2) = p+1$ and $P(1) = 1$ and $P(2) = p$ respectively. (As earlier, for the sake of simplicity, we replace N_n and D_n by $P(n)$.)

Case I : $P(1) = 1$ and $P(2) = p+1$

In this case, substituting $n = 1$ and $n = 2$ in (5) we get, (by Lemma 2 [4]),

$$1 = P(1) = c(\frac{1+\sqrt{5}}{2}) + d(\frac{1-\sqrt{5}}{2})$$

and

$$p+1 = P(2) = c(\frac{1+\sqrt{5}}{2})^2 + d(\frac{1-\sqrt{5}}{2})^2.$$

Solving for c and d, we get

$$c = \frac{p(\sqrt{5}-1)+2}{2\sqrt{5}}$$

and

$$d = \frac{p(\sqrt{5}+1)-2}{2\sqrt{5}}$$

.

Case II : $P(1) = 1$ and $P(2) = p$

As in Case I, substituting $n = 1$ and $n = 2$ in (5) we get, (by Lemma 2 [4]),

$$1 = P(1) = c(\frac{1+\sqrt{5}}{2}) + d(\frac{1-\sqrt{5}}{2})$$

and

$$p = P(2) = c(\frac{1+\sqrt{5}}{2})^2 + d(\frac{1-\sqrt{5}}{2})^2.$$

Solving for c and d, we get

$$c = \frac{p(\sqrt{5}-1)+3-\sqrt{5}}{2\sqrt{5}}$$

and

$$d = \frac{p(\sqrt{5}+1)-3-\sqrt{5}}{2\sqrt{5}}$$

.

□

Observe that for $p = 2$, one gets the Lucas Numbers (defined by the sequence

$$L_n = L_{n-1} + L_{n-2},$$

with $L_1 = 1$, $L_2 = 3$) from the numerator of the n'th convergent whose value is then got from Theorem 7.

Analogous to Binet's form as discussed in the previous section, we get the following result for the numerator and denominator of G.

Corollary 8 *Letting* $\phi = \frac{1+\sqrt{5}}{2}$ *and* $\overline{\phi}$*, its conjugate, the numerator and denominator of* G *from 7 are* $\frac{\phi^n - \overline{\phi}^n}{\phi - \overline{\phi}} + p\frac{-\overline{\phi}^{n-1} + \phi^{n-1}}{\phi - \overline{\phi}}$ *and* $\frac{\phi^n - \overline{\phi}^n}{\phi - \overline{\phi}} + (1-p)\frac{-\phi^{n-1} + \overline{\phi}^{n-1}}{\phi - \overline{\phi}}$ *respectively, where* $\phi\overline{\phi} = -1$.

PROOF.Clear. □

4 An application to physics

The relationship between continued fractions and physics have been long studied as a result of a question first raised by Dwight E. Neuenschwander [5]. In this section, we study an application of continued fractions to physics.

Consider the infinite electrical network with resistances in series and parallel as shown in Figure (1). We can see that the following Lemma holds for such a network.

Lemma 9 *The effective resistance between nodes A and B is the C.F. F. The truncated networks (obtained by considering the first series resistance, the first series and parallel resistance taken together, the first two series resistances taken together with the first resistance* *that is parallel to both, etc) have the effective resistance* p, $p+\frac{1}{p}$, $p+\frac{1}{p+\frac{1}{p}}$ *etc. with the n'th truncated network having the effective resistance between* A *and* B *to be given by* $\frac{N_n}{D_n}$ *from Theorem 4.*

The following corollaries are then immediate.

Corollary 10 *If* $p=1$, *then the effective resistance between the nodes* A *and* B *is the golden mean* $\frac{1+\sqrt{5}}{2}$.

Corollary 11 *If all the resistances are taken to be* 1 *except the first parallel resistance (which is taken to be* $\frac{1}{2}$*), then the effective resistance between* A *and* B *is* $\frac{5+\sqrt{5}}{2}$.

Proof. The C.F. associated with this network is G with $p=2$. Let $z=1+\frac{1}{2+\frac{1}{x}}=\frac{3x+1}{2x+1}$, where $x=\frac{1+\sqrt{5}}{2}$. On simplification, the result follows. □

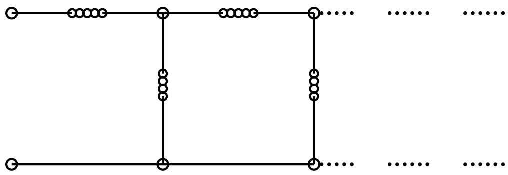

Figure 1: An infinite network of series and parallel resistors with resistance p and $\frac{1}{p}$ respectively.
The top left node is A while the bottom left node is B.

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