2016 HAWAII UNIVERSITY INTERNATIONAL CONFERENCES
SCIENCE, TECHNOLOGY, ENGINEERING, ART, MATH & EDUCATION JUNE 10 - 12, 2016
HAWAII PRINCE HOTEL WAIKIKI, HONOLULU

NOTE ON THE ABEL MATRIX TRANSFORMATIONS

LEMMA, MULATU & ET AL
SAVANNAH STATE UNIVERSITY
COLLEGE OF SCIENCE & TECHNOLOGY
DEPARTMENT OF MATHEMATICS

Note On The Abel Matrix Transformations

Mulatu Lemma, Jonathan Lambright and Latrice Tanksley
College of Science & Technology
Savannah State University
2016 Hawaii University International Conference

Abstract. Let t be sequence in (0,1) that converges to 1. The Abel matrix is defined as a_{nk}= (1-$t_n)^k t_n$. We denote the Abel Matrix by A_t. A_t is a sequence to sequence mapping. When a matrix A_t is applied to a sequence x, we get a new sequence $A_t x$ whose nth term is given by:

$$(A_t x)_n = (1 - t_n) \sum_{k=o}^{\infty} t_n^{\ k} x_k$$

The sequence $A_t x$ is called the A_t -transform of the sequence x.

The purpose of this research is to study the effect of applying A_t to convergent sequences, bounded sequences, divergent sequences, and absolutely convergent sequences by considering the following interesting main questions.

(1) What is the domain of t for which A_t maps convergent sequence into convergent sequence?

(2) What is the domain of t for which the A_t maps absolutely convergent sequence into absolutely convergent sequence?

(3) Does A_t maps unbounded sequence to convergent sequence?

(4) Does A_t maps divergent sequence to convergent sequence?

The strength of the A_t comparing to the identity matric will also be investigated.

Background Materials

w= {the set of all complex sequences}

c= {the set of all convergent complex sequences}

l^{∞} ={the set of all bounded complex sequences)

l = {y: $\sum_{k=0}^{\infty}|y_k| < \infty$}

Definition: A matrix A is an x-y matrix if the image Au of u under the transformation A is in Y wherever u is in x.

Regular Matrix

A matrix is regular if $lim_{n\to\infty}Z_n$= a⇒ $lim_{n\to\infty}(AX)_n$=a. That is a sequence Z is convergent to A⇒ the A-transform of Z also converses to a.

The Sliverman-Toeplitz Thoerem

We state the following famous Sliverman-Toeplitz Theorem as Proposition I with out proof and apply it.

Proposition I: A matrix A $=(a_{n,k})$ is regular if and only if

(i) $\lim_{n\to\infty} a_{n,k} = 0$ for each k= 0,1,...,

(ii) $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k} = 1$, and

(iii) $\sup_n\{ \sum_{k=0}^{\infty} |a_{n;k}|\} \leq M < \infty$ for some $M < 0.$

The Main Results

Theorem: The Abel Matrix A_t is a regular matrix

Proof: We use proposition 1 to prove the theorem. Note that

(1) $\lim_{n\to\infty} a_{n,k} = \lim_{n\to\infty} (1-t_n)^k t_n = 0$

(2) $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = \lim_{n\to\infty} \sum_{k=0}^{\infty} t_n{}^k (1-t_n) = \lim_{n\leftarrow\infty} (1-t_n) \sum_{k=0}^{\infty} t_n{}^k =$

$\frac{1-t_n}{1-t_n} = 1$ and

(3) $Sup_n \sum_{k=0}^{\infty} a_{n,k} = 1$

Hence by Proposition I, the Abel Matrix A_t is a regular matrix

Remark 1: Does the A_t matrix map a bounded sequence into a convergent sequence?

The answer is affirmative as shown by the following example.

Example 1: Consider the bounded sequence given by $x_k = (-1)^k$

Then $(A_t x)_n = (1-t_n) \sum_{k=0}^{\infty} (t_n)^k (-1)^k$

$$= (1-t_n) \sum_{k=0}^{\infty} (-t^n)^k$$

$$= (1-t_n) \frac{1}{1+t_n}$$

Knopp Thorem

The Matrix A is an ℓ - ℓ matrix if and only if there exists a number $M > 0$ such that for every k,

$$\sum_{n=0}^{\infty}|a_{nk}| \le M.$$

Theorem 2: $M_g is \;\; \ell - \ell \Leftrightarrow (1-g) \in \ell$

Lemma 1:

$$M_g is \;\; \ell - \ell \;\; \Rightarrow \;\; (1-g) \in \ell \; .$$

Proof: We use the Knopp-Lorentz Rule:

$$M_g is \;\; \ell - \ell \;\; \Rightarrow \;\; \sum_{n=0}^{\infty}\left|(1-g_n)g_n^k\right| \le M$$

$$\Rightarrow \;\; \sum_{n=0}^{\infty}\left|(1-g_n)\right| \le M \quad \text{(for k=0)}$$

$$\Rightarrow \;\; (1-g) \in \ell$$

Lemma 2:

$$1-g \in \ell \;\; \Rightarrow \;\; M_g isl-l$$

Proof: We use the Knopp-Lorentz Rule:

$$\sum_{n=0}^{\infty}|a_{nk}| \le \sum_{n=0}^{\infty}\left|(1-g_n)g_n^k\right|$$

$$\le \sum_{n=0}^{\infty}(1-g_n) \le M \quad \text{for some M>0 as} \quad (1-g) \in \ell \; .$$

Now Theorem 2 follows by Lemmas 1&2.

Corollary 1. *If* $< 0 < g_n < w_n < 1$, *then* M_g *is an* $\ell - \ell$ *matrix whenever* G_t *is an* $\ell - \ell$ *matrix.*

Proof: M_g *is* $\ell - \ell \Rightarrow (1\text{-}g) \in l$ *by theorem 2.*

Now $< t_n < w_n \Rightarrow (1\text{-}w_n) \le (1\text{-}g_n)$ *and* the corollary follows by Theorem 2.

Corollary 2. arcsin(1−g) $\in$ l $\Leftrightarrow$ M $_g$ is an l-l matrix.

Proof: The corollary easily follows using Theorem 2 and the following basic inequality.

$$(1\text{-g}) \le \arcsin(1\text{-g}) \le \frac{(1-g)}{\sqrt{1-(1-g)^2}}.$$

Corollary 3. Let w_n = 1/g $_n$. Then the zeta matrix $_n$ [2], is l-l whenever M $_g$ is an l-l matrix.

Theorem 3 $\frac{-1}{\ln(1-g_n)} \in$ l $\Rightarrow$ $\mathbf{M}_g$ is an l-l matrix.

Proof. Note that

$$1\text{-g}_n := \left(\sum_{k=0}^{\infty} g_n{}^k\right)^{-1}$$

$$\le \left(\sum_{k=0}^{\infty} \frac{1}{k+1} g_n{}^k\right)^{-1}$$

$$= \left(\sum_{k=0}^{\infty} g_n{}^k \left(\int_0^1 V^k dV\right)\right)^{-1}$$

$$= \left(\sum_{k=0}^{\infty} \left(\int_0^1 g_n{}^k V^k dV\right)\right)^{-1}$$

$$= \left(\int_0^1 dV \left(\sum_{k=0}^{\infty} (g_n V)^k\right)\right)^{-1}$$

The Interchanging of the Integral and summation is legitimate as the power series

$\sum_{k=0}^{\infty} (Vg_n)^k$ converges absolutely and uniformly for $0 \le Vg_n \le$ 1. Hence we have,

$$1\text{-g}_n \le \left(\int_0^1 \frac{dV}{1-Vg_n}\right)^{-1}$$

$$= \left(\frac{-1}{t_n}(\ln(1-g_n)\right)^{-1}$$

$$\leq \frac{-1}{\ln(1-g_n)}$$

The hypothesis that $\frac{-1}{\ln(1-g)} \in$ l$\Rightarrow$(1-g)$\in$l and hence by Theorem 2, M_g is l-l.

Remark 3. An l-l Mg matrix maps a bounded sequence into l as shown by the following example. This shows that the M_g matrix is stronger than the identity matrix in the *l-l* setting or l(A) is larger than *l*.

Example 3.

Assume M_g matrix is an *l-l* and consider the bounded sequence given by

$x_k = (-1)^k$

Then
$$(M_g x)_n = (1-g_n)\sum_{k=0}^{\infty}(g_n)^k(-1)^k$$
$$= (1-g_n)\sum_{k=0}^{\infty}(-g^n)^k$$
$$= (1-g_n)\frac{1}{1+g_n}$$
$$\leq (1-g_n)$$

Now M_g matrix is l-*l* $\Rightarrow$ *(1-g)*$\in$l , by Theorem 2, and hence $M_g x \in l.$

Remark 4: An l-l M_g matrix maps unbounded sequence into l as shown by the following example.

Example 4: Assume M_g matrix is an *l-l* and consider the unbounded sequence given by

$x_k = (-1)^k(k+1)$. Note that

$$(M_g x)_n = \sum_{k=0}^{\infty} (1-g_n) g_n^k (-1)^k (k+1)$$

$$= (1-g_n) \sum_{k=0}^{\infty} g_n^k (-1)^k (k+1)$$

$$= (1-g_n) \sum_{k=0}^{\infty} (-g_n)^k (k+1)$$

$$= \frac{1-g_n}{(1+g_n)^2}$$

$$\leq (1-g_n)$$

Now M_g matrix is l-*l* $\Rightarrow$ *(1-g)*$\in$ l , by Theorem 2, and hence $M_g x \in l.$

References:

1. M. Lemma, *Logarithmic Transformations into* l1, Rocky Mountain J. Math **28** (1998), no. 1,253–266.

2. L. K. Chu, *Summability methods based on the Riemann zeta function*, Internat. J. Math. Math. Sci. **11** (1988), no. 1, 27–36

3.. E. Powell and S. M. Shah, *Summability Theory and Applications*, Prentice-Hall of India,New Delhi, 1988..

$\lim_{n\to\infty}(A_t x)_n = 0$; hence $A_t x$ Î c

Remark 2: Does the A_t matrix map anunbounded divergent sequence x, into a convergent sequence?

The answer is affirmative as shown by the following example.

Example 2: Consider the unbounded sequence given by $x_k = (-1)^k(k+1)$. Note that

$$(A_t x)_n = \sum_{k=0}^{\infty}(1-t_n)t_n^k(-1)^k(k+1)$$

$$= (1-t_n)\sum_{k=0}^{\infty}t_n^k(-1)^k(k+1)$$

$$= (1-t_n)\sum_{k=0}^{\infty}(-t_n)^k(k+1)$$

$$= \frac{1-t_n}{(1+t_n)^2}$$

Now, $\lim_{n\to\infty}(A_t x)_n = \lim_{n\to\infty}\frac{1-t_n}{(1+t_n)^2} = 0$

Hence $A_t x \in C.$