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THE MATHEMATICAL BEAUTY OF GEOMETRICAL POWER SERIES

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The Mathematical Beauty of Geometric Power Series

Scholarship of Teaching and Learning Paper

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Abstract: The process of adding infinitely many numbers is at the heart of the mathematical concept of a numerical series. One of the keys to adding infinitely many such numbers is to use the Geometric Power series. The Geometric Power Series is series of the form $\sum_{k=0}^{\infty} ax^k$. The purpose of this study is to investigate some important properties and applications of the geometric power series $\sum_{k=0}^{\infty} x^k$ in $|x|<1$. The Geometric Power Series is the simplest and most applicable power series. It is commonly used in differential equations, physics, and engineering. We will see that this power series has surprisingly many applications.

Introduction

Power series representation of functions has played an important role in the development of calculus. In fact, much of Newton's work with differentiation and integration was done in the context of power series-especially his work with complicated algebraic functions and transcendental functions. Euler, Lagrange, Leibniz, and the Bernoullis all used power series exclusively in calculus.

The representation of functions by power series is one of the most useful of mathematical techniques in a wide variety of situations. Sometimes we start from a function that is defined for us in some manner not employing series, and seek to expand the function in a series. At other times we may form a power series, or have one represented to us, and then undertake to use this function in some way. In either of these situations we need to know something of what properties a function has if it is defined by a power series.

Let f(x) be the function represented by the series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

Then f(x) is called a power series function.

More generally, if f(x) is represented by the series

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

Then we call f(x) a power series centered at x=c. the domain of f(x) is called the Interval of Convergence and half the length of the domain is call the radius of Convergence

A Geometric Power Series is a power series given as

$$\text{f(x)} \quad = \sum_{n=0}^{\infty} ax^n$$

Background Materials

We use the following concepts and results in the paper and we state them for future reference.

1. Once a function is given as a power series, it is continuous wherever it converges and is differentiable on the interior of this set.

2. The power series can be differentiated and integrated quite easily, by treating every term separately:

$$f'(x) = \sum_{n=1}^{\infty} a_n n \left(x - c\right)^{n-1}$$

$$\int f(x)\,dx = \sum_{n=0}^{\infty} \frac{a_n \left(x - c\right)^{n+1}}{n+1} + C$$

3. The radius of convergence of a power series is a non-negative quantity— either a real number or +∞—that represents a range (within the radius) in which the function will converge.

For a power series *f* defined as:

$$f(z) = \sum_{n=0}^{\infty} f_n = \sum_{n=0}^{\infty} c_n (z-a)^n,$$

where

a is a constant, the center of the circle of convergence,
c_n is the nth complex coefficient (note that real numbers are a very common special case of complex numbers),
z is a variable and
f_n is the nth term of the series

4. The radius of convergence r is a nonnegative real number or ∞, such that the series converges if

$$|z - a| < r$$

and diverges if

$$|z - a| > r.$$

In other words, the series converges if z is close enough to the center and diverges if it is too far away. The radius of convergence specifies how close is close enough. The radius of convergence is infinite if the series converges for all complex numbers z.

5. We use the ratio test to find the interval where the series is absolutely convergent.

6. To obtain the derivative or the integral of a power series function, we can pass the derivative or integral through $\sum$

7. By the Geometric Series Test this series converges for $|x| < 1$, hence the center of convergence is 0 and the radius is1.

The Main Results

Lemma 1: If -1<x<1 and $S_n = \sum_{k=0}^{\infty} x^k$ then $S_n = \frac{1-x^{n+1}}{1-x}$

Proof:
$S_n = 1 + x + x^2 + ... + x^n$ and $xS_n = x + x^2 + x^3 + x^n + x^{n+1}$

Now,

$$S_n - xS_n = 1 - x^{n+1}$$

$$S_n(1-x) = \frac{1-x^{n+1}}{1-x}$$

$$S_n = \frac{1-x^{n+1}}{1-x}$$

Remark I: If $|x| < 1$, the $\lim_{n\to\infty} x^n = 0$

Theorem 1: If -1<x<1, then $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$

Proof: The result easily follows by Lemma I and Remark I as

$\lim_{n\to\infty}(\sum_{k=0}^{n} x^k) = \lim_{n\to\infty}(\frac{1-x^{n+1}}{1-x})$ implies that $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$

Some Problem Solving Techniques using Theorem 1

Example 1:

Evaluate: $(\sum_{k=1}^{\infty} \frac{3^k + 4^k}{6^k})$

Solution:

$$= \sum_{k=1}^{\infty} (\frac{3^k}{6^k} + \frac{4^k}{6^k})$$

$$= \sum_{k=1}^{\infty} (\frac{1}{2})^k + (\frac{2}{3})^k$$

$$= \sum_{k=1}^{\infty} (\frac{1}{2})^k + \sum_{k=1}^{\infty} (\frac{2}{3})^k$$

$$= \sum_{k=0}^{\infty} (\frac{1}{2})^{k+1} (+) \sum_{k=0}^{\infty} (\frac{1}{2})^{k+1}$$

$$= \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)(1) + \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k \left(\frac{2}{3}\right)(1)$$

$$= \frac{1}{2} \sum_{k=0}^{\infty} (\frac{1}{2})^k + \frac{2}{3} \sum_{k=0}^{\infty} (\frac{2}{3})^k$$

$$= \frac{1}{2}\left(\frac{1}{1 - 1/2}\right) + \frac{2}{3}\left(\frac{1}{1 - 2/3}\right)$$

$$= \frac{1}{2}(2) + \frac{2}{3}(3)$$

$$= 1 + 2$$

$$= 3$$

Example 2:

Evaluate: $\sum_{k=2}^{\infty}(-1)(2)^{-2k}$

Solution: $=\sum_{k=2}^{\infty}(-1)(2)^{-2k}$

$$=\sum_{k=2}^{\infty}\left(\frac{-1}{2}\right)^{2k}$$

$$=\sum_{k=2}^{\infty}\left(\frac{1}{4}\right)^{k}$$

$$=\sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^{k+2}$$

$$=\sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^{k}\left(\frac{1}{4}\right)^{2}$$

$$=\frac{1}{16}\sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^{k}$$

$$=\frac{1}{16}\left(\frac{1}{1-1/4}\right)$$

$$=\frac{1}{16}\left(\frac{4}{3}\right)$$

$$=\frac{1}{12}$$

Example 3:

Evaluate: $\dfrac{17+15+48+24+12+6...}{16+8+4+2...}$

Solution:

$$= \frac{17+15+48+24+12+6...}{16+8+4+2...}$$

$$= \frac{17+15+48\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\right)...}{16\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\right)...}$$

$$= \frac{32+48\left(\sum_{k=0}^{\infty}(\frac{1}{2})^{k}\right)}{16\left(\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^{k}\right)}$$

$$= \frac{32+48\left(\frac{1}{1-\frac{1}{2}}\right)}{16\left(\frac{1}{1-\frac{1}{2}}\right)}$$

$$= \frac{32+48(2)}{16(2)}$$

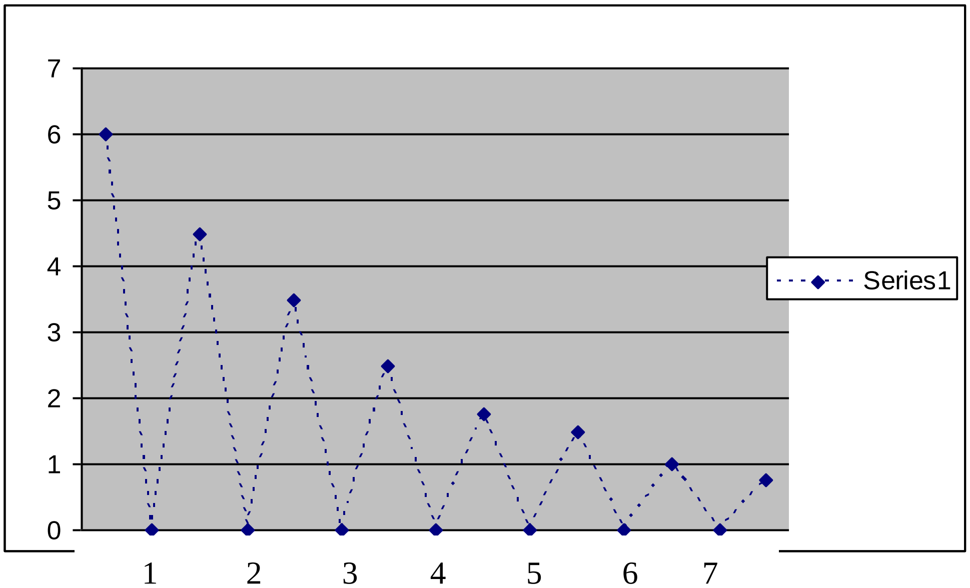
$$= \frac{32+96}{32}$$

$$= \frac{148}{32}$$

$$= 4$$

Example 4: Application to Physics (Bouncing Ball Problem)

A ball is dropped from a height of 6 feet and begins bouncing, as shown below. The height of each bounce is three-fourths the height of the previous bounce. Find the total vertical distance traveled by the ball.



Solution

When the ball hits the ground for the first time, it has traveled a distance of $D_1 = 6$ feet. For subsequent bounces, let D_i be the distance traveled up and down. For example, D_2 and D_3 are as follows.

$$D_2 = 6(\frac{3}{4}) + 6(\frac{3}{4}) = 12(\frac{3}{4})$$

$$D_3 = 6(\frac{3}{4})(\frac{3}{4}) + 6(\frac{3}{4})(\frac{3}{4}) = 12(\frac{3}{4})^2$$

By continuing this process it can be determined that the total vertical distance is

$$D = 6 + 12(\frac{3}{4}) + 12(\frac{3}{4})^2 + 12(\frac{3}{4})^3 + ...$$

$$= 6 + 12\sum_{n=0}^{\infty}(\frac{3}{4})^{n+1}$$

$$= 6 + 12(\frac{3}{4})\sum_{n=0}^{\infty}(\frac{3}{4})^{n}$$

$$= 6 + 9\left(\frac{1}{1-\frac{3}{4}}\right)$$

$$= 6 + 9(4)$$
$$= 42 Feet$$

Example 5: Find the Power Series Expansion of $f(x) = \frac{1}{2-x}$

$$f(x) = \frac{1}{2-x} = \frac{1}{2(1-x/2)} = \frac{1}{2}\left(\frac{1}{1-x/2}\right) = \frac{1}{2}\sum_{k=0}^{\infty}(x/2)^k = \sum_{k=0}^{\infty}\frac{x^k}{2^{k+1}} \;(|x| < 2)$$

Geometric Power Series and Derivatives

Lemma 2:

If $|x| < 1$ and $f(x) = \frac{1}{1-x}$, then $f^{(n)}(n) = \frac{n!}{(1-x)^{n+1}}$

Proof: $f(x) = \sum_{k=0}^{\infty} x^k$. Then

$$f^{(I)}(x) = \frac{1}{(1-x)^2} = \frac{1!}{(1-x)^{1+1}}$$

$$f^{(II)}(x) = \frac{2}{(1-x)^3} = \frac{2!}{(1-x)^{2+1}}$$

$$f^{(III)}(x) = \frac{6}{(1-x)^4} = \frac{3!}{(1-x)^{3+1}}$$

$$f^{(IV)}(x) = \frac{24}{(1-x)^5} = \frac{4!}{(1-x)^{4+1}}$$

$$f^{(v)}(x) = \frac{120}{(1-x)^6} = \frac{5!}{(1-x)^{5+1}}$$

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$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$$

Lemma 3:

Let -1<x<1 and f(x) = $\sum_{k=0}^{\infty} x^k$. Then $\frac{f^{n}(x)}{n!} = \sum_{k=0}^{\infty} \binom{k+n}{k} x^k$.

Proof: $f(x) = \sum_{k=0}^{\infty} x^k$

$$f^{I}(x)=\sum_{k=1}^{\infty}(k)x^{k-1}=\sum_{k=0}^{\infty}(k+1)x^{k}\Rightarrow\frac{f'(x)}{1!}=\sum_{k=0}^{\infty}\binom{k+1}{k}x^{k}$$

$$f^{II}(x)=\sum_{k=1}^{\infty}(k+1)kx^{k-1}=\sum_{k=0}^{\infty}(k+2)(k+1)x^{k}\Rightarrow\frac{f''(x)}{2!}=\sum_{k=0}^{\infty}\binom{k+2}{k}x^{k}$$

$$f^{III}(x)=\sum_{k=1}^{\infty}(k+2)(k+1)kx^{k-1}=\sum_{k=0}^{\infty}(k+3)(k+2)(k+1)x^{k}\Rightarrow\frac{f'''(x)}{3!}=\sum_{k=0}^{\infty}\binom{k+3}{k}x^{k}$$

$$f^{IV}(x)=\sum_{k=1}^{\infty}(k+3)(k+2)(k+1)kx^{k-1}=\sum_{k=0}^{\infty}(k+4)(k+3)(k+2)(k+1)x^{k}\Rightarrow\frac{f^{IV}(x)}{4!}=\sum_{k=0}^{\infty}\binom{k+4}{k}x^{k}$$

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The above pattern $\Rightarrow\frac{f^{n}(x)}{n}=\sum_{k=0}^{\infty}\binom{k+n}{k}(x)^{k}$.

Theorem 2

If $\sum_{k=0}^{\infty}x^{k}=\frac{1}{1-x},-1<x<1, then \sum_{k=0}^{\infty}\binom{k+n}{k}x^{k}=\frac{n!}{(1-x)^{n+1}}$

Proof:

The Theorem easily follows by Lemma 2 and 3 as $\frac{d^{n}}{dx^{n}}(\sum_{k=0}^{\infty}x^{k})=\frac{d^{n}}{dx^{n}}(\frac{1}{1-x})$

Remark II: Note that the radius of convergence of $\sum_{k=0}^{\infty}x^{n}$ and its derivative is 1.

<u>Some Problem Solving Techniques Using Theorem 2.</u>

Example 6. Find the exact sum of $\sum_{k=2}^{\infty}(k-1)(k)(\frac{1}{2})^{k}$

Solution:

Note that by Theorem 2,

$$\sum_{k=0}^{\infty}(k+2)(k+1)x^k = \frac{2!}{(1-x)^3}$$

We have,

$$\sum_{k=2}^{\infty}(k-1)(k)(x)^k = \sum^{\infty}(k+2)(k+1)x^{k+2} = \frac{2!}{(1-x)^3}$$

$$\Rightarrow \sum_{k=0}^{\infty}(k+2)(k+1)(\frac{1}{2})^{k+2} = \frac{2!}{(1-\frac{1}{2})^3}$$

$$\frac{1}{4}\sum_{k=0}^{\infty}(k+2)(k+1)(\frac{1}{2})^k = \frac{2!}{(1-\frac{1}{2})^3}$$

$$\sum_{k=0}^{\infty}(k+2)(k+1)(\frac{1}{2})^k = 4(\frac{2}{1/8})$$

$$\sum_{k=0}^{\infty}(k+2)(k+1)(\frac{1}{2})^k = 64$$

Example 7. Find the exact sum of:

$$\sum_{k=0}^{\infty}(-1)^k(k+1)(2^{-2k}+3^{-k})$$

Solution

$$\sum_{k=0}^{\infty}(-1)^k(k+1)(2^{-2k}+3^{-k})$$

$$= \sum_{k=0}^{\infty}(-1)^k(k+1)(2)^{-2k} + \sum_{k=0}^{\infty}(-1)^k(k+1)(3)^{-k}$$

$$= \sum^{\infty}(k+1)(\frac{-1}{4})^k + \sum_{k=0}^{\infty}(k+1)(-\frac{1}{3})^k$$

Note that by Theorem 2, $\sum_{k=0}^{\infty}(k+1)x^k = \frac{1!}{(1-x)^2}$.

$$\Rightarrow \sum_{k=0}^{\infty}(k+1)(-\frac{1}{4})^{k}+\sum_{k=0}^{\infty}(k+1)(-\frac{1}{3})^{k}=\frac{1!}{(1+\frac{1}{4})^{2}}+\frac{1!}{(1+\frac{1}{3})^{2}}$$

$$\sum_{k=0}^{\infty}(k+1)(-\frac{1}{4})^{k}+\sum_{k=0}^{\infty}(k+1)(-\frac{1}{3})=\frac{1}{(\frac{5}{4})^{2}}+\frac{1}{(\frac{4}{3})^{2}}$$

$$\sum_{k=0}^{\infty}(k+1)(-\frac{1}{4})+\sum_{k=0}^{\infty}(k+1)(-\frac{1}{3})^{k}=\frac{481}{400}$$

Example:8

Find the power series expansion of f(x)= $\frac{1}{x^{2}}$ using Theorem 2.

Solution:
Note that,

$$f(x)=\frac{1}{(1-(1-x)^{2}}=\sum_{k=0}^{\infty}(k+1)(1-x)^{k}$$

Geometric Series and Integration

1. If $\sum_{k=0}^{\infty}x^{k}=\frac{1}{1-x}$ $|x|<1$, then $\int\left(\sum_{k=0}^{\infty}x^{k}\right)dx=\int\frac{dx}{1-x}\Rightarrow\sum_{k=0}^{\infty}\frac{x^{k+1}}{k+1}=\ln(1-x)+c$

Note if x=0, we have c=0 and $\sum_{k=0}^{\infty}\frac{x^{k+1}}{k+1}=-\ln(1-x)$

Example 9:

Evaluate: $\sum_{k=0}^{\infty}\frac{1}{k+1}\left(\frac{1}{2}\right)^{k}$.

Solution:

Note: $\sum_{k=0}^{\infty}\frac{1}{k+1}(x)^{k+1}=-\ln(1-x),-1<x<1$

$$\sum_{k=0}^{\infty}\frac{1}{k+1}\left(\frac{1}{2}\right)^{k+1} = \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{k+1}\left(\frac{1}{2}\right)^{k} = -\ln\left(\frac{1}{2}\right) = \sum_{k=0}^{\infty}\frac{1}{k+1}\left(\frac{1}{2}\right)^{k} = 2\ln 2$$

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